UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2007

B.Sc. III/B.Ed./B.A.S.S. III

TITLE OF PAPER: ABSTRACT ALGEBRA

COURSE NUMBER: M323

TIME ALLOWED: THREE HOURS

INSTRUCTIONS:

1. This paper consists of SEVEN questions on FOUR pages.
2. Answer any FIVE questions.
3. Non-programmable calculators may be us

SPECIAL REQUIREMENTS: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

Question 1

(a) Find the number of elements in each of the cyclic subgroups
(i) $< 30 >$ of $\mathbb{Z}_{42}$

[3 marks]

(ii) $< 15 >$ of $\mathbb{Z}_{48}$

[3 marks]

(b) For $\mathbb{Z}_{12}$, find all the subgroups and give a lattice diagram

[7 marks]

(c) (i) Find all cosets of $H = \{0, 6, 12\}$ in $\mathbb{Z}_{18}$

[4 marks]

(ii) Show that $\mathbb{Z}_6$ and S_3 are not isomorphic

[3 marks]

Question 2

(a) Prove that every finite group of prime order is cyclic

[5 marks]

(b) Show that the set $G = \mathbb{Q} - \{0\}$ with respect to the operation

$$a * b = \frac{ab}{10}$$

is a group

[9 marks]

(c) Prove that if $(ab)^{-1} = a^{-1}b^{-1} \ \forall a, b \in G$, where G is a group then G is abelian.

[6 marks]

Question 3

(a) If $\varphi : G \longrightarrow H$ is an isomorphism of groups and e is the identity of G then
(i) $(e)\varphi$ is the identity element in H
(ii) $(a^n)\varphi = [(a)\varphi]^n \quad \forall n \in \mathbb{Z}^+$

[12 marks]

(b) (i) State Lagrange's theorem

[2 marks]

(ii) Using b(i) above or otherwise, show that $\mathbb{Z}_p$ has no proper subgroups if p is a prime number

[6 marks]

Question 4

(a) Prove that if $(a, s) = 1$ and $(b, s) = 1$ then $(ab, s) = 1 \quad \forall a, b, s \in \mathbb{Z}$

[6 marks]

(b) Give a single numerical example to **disprove** the following;
"If $ax \equiv bx \pmod{n}$ then $a \equiv b \pmod{n} \quad a, b, n \in \mathbb{Z}$"

[4 marks]

(c) Prove that every cyclic subgroup of a cyclic group is cyclic

[10 marks]

Question 5

(a) Solve the following system

$$\begin{aligned} 2x &\equiv 1 (\text{mod } 5) \\ 3x &\equiv 4 (\text{mod } 7) \end{aligned}$$

[7 marks]

(b) Find the number of generators of cyclic groups of order 8 and 60

[7 marks]

(c) Prove that a non-abelian group of order $2p, p$ prime contains at least one element of order p.

[6 marks]

Question 6

(a) Prove that every cyclic group is abelian.

[5 marks]

(b) Let n be a positive integer greater than 1 and let, for $a, b, \in \mathbb{Z}$

$$a \text{ R } B \Leftrightarrow a \equiv b \ (\text{mod } n)$$

Show that R is an equivalence relation on $\mathbb{Z}$.

[7 marks]

(c) Let H be the subset of $\{\rho_0 = (1), \rho_1 = (123), \rho_2 = (132)\}$ of the symmetric group S_3

(i) Show that H is a subgroup of S_3

(ii) Show that H is a cyclic

[8 marks]

Question 7

(a) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 3 & 2 & 7 & 8 & 6 & 5 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 7 & 2 & 3 & 8 & 5 & 6 \end{pmatrix}$

(i) Express α and β as products of disjoint cycles and then as products of transpositions. For each permutation say whether it is an even permutation or an odd one.

[8 marks]

(ii) Compute $\alpha^{-1}, \beta^{-1}\alpha, (\alpha\beta)^{-1}$

[6 marks]

(b) Find the greatest common divisor d of the number 204 and 54 and express it in the form

$$d = 204x + 54y$$

for some $x, y \in \mathbb{Z}$

[6 marks]

******** *END OF EXAMINATION* ********