UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2007

B.Sc. / B.Ed. / B.A.S.S. III

TITLE OF PAPER	:	Real Analysis
COURSE NUMBER	:	M331
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Let S be a nonempty subset of $\mathbb{R}$.

(i) Define $sup\, S$; [2]

(ii) Prove that if S has an upper bound, then it has a supremum. [12]

(b) Let S be the set $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$. Show that $inf S = 0$ and $sup\, S = 1$. [6]

QUESTION 2

(a) (i) Prove that every sequence of real numbers has a monotone subsequence. [8]

(ii) Hence, or otherwise, prove the Bolzano Weiestrass theorem. [4]

(b) (i) What is a monotone sequence? [2]

(ii) Show that the sequence defined by the recurrence relation

$$a_1 = \sqrt{2}, \qquad a_n = \sqrt{a_{n-1} + 2} \quad \text{for all } n = 2, 3, 4, \ldots$$

is convergent, and find its limit. [6]

QUESTION 3

(a) Let A be a nonempty bounded set of real numbers. Prove that if $inf A$ does not belong to A, then there exists a decreasing sequence $\{a_n\} \subset A$ such that $inf A < a_n$ for all $n \in \mathbb{N}$. [8]

(b) (i) Define the *limit superior* and the *limit inferior* of a bounded sequence of real numbers. [3]

(ii) Using the definition in (i), find

$$\lim_{n\to\infty} sup \{-2\frac{1}{2}, 0, 1, 3\frac{1}{2}, -2\frac{1}{3}, 0, 1, 3\frac{1}{3}, -2\frac{1}{4}, 0, 1, 3\frac{1}{4}, \ldots\}.$$

[3]

(c) Let A and B be nonempty subsets of $\mathbb{R}$, and define $A + B = \{a + b : a \in A, b \in B\}$. Prove or disprove that if A and B are bounded below, then $inf(A + B) = inf\, A + inf B$. [6]

QUESTION 4

(a) Find $\lim_{x\to c} f(x)$ for each of the following functions and values of c:

(i)

$$f(x) = \begin{cases} \frac{x^2-25}{x-5} & \text{if } x \neq 5 \\ 4 & \text{if } x = 5, \end{cases}$$

(ii)

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

[6]

(b) (i) What is meant by saying that a function $f(x)$ is *continuous* at a point c? (You may assume that f is defined on an interval (a, b) that contains c).

(ii) Show that if $f(x)$ and $g(x)$ are both continuous at c, then so is their product. [7]

(c) Which of the following functions is continuous at the point 0? Give reasons for your answers.

(i)

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

(ii) $f(x) = [x]$, where $[x]$ is the integer part of x. [7]

QUESTION 5

(a) Let $f : [a, b] \longrightarrow \mathbb{R}$ and let $c \in (a, b)$. What is meant by saying that f is *differentiable* at the point c? Show that if f is differentiable at c, then f is continuous at c. [7]

b State and prove Rolle's Theorem. Use Rolle's theorem to deduce the Mean Value Theorem for a differentiable function $f(x)$ defined on an interval $[a, b]$.[7]

(c) Use the Mean Value Theorem to show that $|\sin^2 x - \sin^2 y| \leq 2|x - y|$ for all $x, y \in \mathbb{R}$. [6]

QUESTION 6

(a) Let $f : [a, b] \longrightarrow \mathbb{R}$. Explain how the *Riemann integral* $\int_a^b dx$ is defined using upper and lower sums. [10]

(b) By considering the integral $\displaystyle\int_1^n \frac{1}{x^2}\, dx$ as $n \to \infty$, show that the series $\displaystyle\sum \frac{1}{n^2}$ is convergent. [10]

QUESTION 7

(a) Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and $F(x) = \int_a^x f(t)\, dt + c$ for all $a \leq x \leq b$. Show that $F(x)$ is differentiable in the interval $[a, b]$, with derivative $DF(x) = f(x)$. [10]

(b) Let $g(x) = \int_0^{x^2} \ln(1 + 1/2 \sin t)\, dt$. Show that g is differentiable for all x and find its derivative. [10]

END OF EXAMINATION