# UNIVERSITY OF SWAZILAND

## SUPPLEMENTARY EXAMINATIONS 2007

## B.Sc. / B.Ed. / B.A.S.S. III

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| TITLE OF PAPER | : | REAL ANALYSIS |
| COURSE NUMBER | : | M331 |
| TIME ALLOWED | : | THREE (3) HOURS |
| INSTRUCTIONS | : | 1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.<br>2. ANSWER ANY FIVE QUESTIONS |
| SPECIAL REQUIREMENTS | : | NONE |

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

(a) Let $S$ be a nonempty subset of $\mathbb{R}$.

(i) Define $inf S$; [2]

(ii) Prove that if $S$ has a lower bound, then it has an infimum. [12]

(b) Let $S$ be the set $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$. Show that $inf S = 0$ and $sup\, S = 1$. [6]

## QUESTION 2

(a) (i) Define what is meant by saying that a sequence $\{a_n\}$ converges to a limit $l$.

(ii) Using the definition, show that $\lim_{n \to \infty} \frac{n+2}{2n+4} = \frac{1}{2}$ [6]

(b) Prove that a nonempty subset of $\mathbb{R}$ which is bounded below can contain at most one of its lower bounds. [3]

(c) Let $S$ be the set $S = \{x \in \mathbb{Q} : x^2 < 2\}$. Show that $inf S = -\sqrt{2}$. [5]

(d) Prove that if a sequence of real numbers is convergent, then its limit is unique. [6]

## QUESTION 3

(a) Which of the following sequences are convergent? For convergent sequences, find the limit (state clearly any facts about limits that you use).

(i) $a_n = \dfrac{3n^2 - n + 3}{5n^2 - 13}$

(ii) $a_n = \sqrt{n + \dfrac{1}{n}}$. [6]

(b) Let $A$ and $B$ be nonempty subsets of $\mathbb{R}$. Define $A + B = \{a + b : a \in A, b \in B\}$ and $cA = \{ca : a \in A\}$, where $c$ is a real number. Prove or disprove the following:

(i) If $A$ and $B$ are bounded below, then $inf(A+B) = inf\, A + inf B$.

(ii) If $c < 0$, then $sup\,(cA) = c\, inf A$. [14]

## QUESTION 4

(a) Find $\lim_{x \to c} f(x)$ for each of the following functions and values of $c$:

(i)

$$f(x) = \begin{cases} \frac{x^4-9}{x^2-3} & \text{if } x^2 \neq 3 \\ 12 & \text{if } x^2 = 3; \end{cases}$$

(ii)

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ -1 & \text{if } x = 0. \end{cases}$$

[6]

(b) (i) What is meant by saying that a function $f(x)$ is *continuous* at a point $c$? (You may assume that $f$ is defined on an interval $(a, b)$ that contains $c$).

(ii) Show that if $f(x)$ and $g(x)$ are both continuous at $c$, then so is their product. [7]

(c) Which of the following functions is continuous at the point 0? Give reasons for your answers.

(i)

$$f(x) = \begin{cases} x \cos \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0; \end{cases}$$

(ii) $f(x) = [x^2]$, where $[z]$ is the integer part of $x$. [7]

## QUESTION 5

(a) Let $f : [a, b] \longrightarrow \mathbb{R}$ and let $c \in (a, b)$. What is meant by saying that $f$ is *differentiable* at the point $c$? Show that if $f$ is differentiable at $c$, then $f$ is continuous at $c$. [7]

(b) State and prove Rolle's Theorem. Use Rolle's theorem to deduce the Mean Value Theorem for a differentiable function $f(x)$ defined on an interval $[a, b]$.[7]

(c) Use the Mean Value Theorem to show that $|\exp x - \exp y| \leq e|x - y|$ for all $x, y \in [0, 1]$, where $e = \exp 1$. [6]

## QUESTION 6

(a) Let $f : [a, b] \longrightarrow \mathbb{R}$. Explain how the *Riemann integral* $\int_a^b dx$ is defined using upper and lower sums. [10]

(b) By considering the integral $\displaystyle\int_1^n \frac{1}{x^2}\, dx$ as $n \to \infty$, show that the series $\sum \frac{1}{n^2}$ is convergent. [10]

## QUESTION 7

(a) Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and $F(x) = \int_a^x f(t)\, dt + c$ for all $a \leq x \leq b$. Show that $F(x)$ is differentiable in the interval $[a, b]$, with derivative $DF(x) = f(x)$. [10]

(b) Let $g(x) = \int_0^{x^3} \exp(1 + 2\sin t)\, dt$. Show that $g$ is differentiable for all $x$ and find its derivative. [10]

END OF EXAMINATION