UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2006/7

BSc. / BEd. / B.A.S.S. III

TITLE OF PAPER	:	DYNAMICS II
COURSE NUMBER	:	M 355
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Use the Lagrangian formulation to find the equations of motion for a system with kinetic and potential energy given by

$$\begin{aligned} T &= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_1(a^2\dot{\theta}^2 + 2a\dot{x}\dot{\theta}\cos\theta) \\ V &= kx^2 + m_1 ga(1 - \cos\theta) \end{aligned}$$

where a, k, g, m_1 and m_2 are constants. [14 marks]

(b) Given that the Lagrangian of a system is given by

$$L = \frac{m}{2}\dot{q}^2 - \frac{1}{2}kq^2 + \lambda q\dot{q}$$

where k, m and λ are constants. Show that the Hamiltonian function corresponding to this Lagrangian is

$$H = \frac{(p - \lambda q)^2}{2m} + \frac{1}{2}kq^2$$

[6 marks]

QUESTION 2

2. The Lagrangian function for a double pendulum is given by

$$L = \frac{1}{2}mb^2(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + mgb(2\cos\theta_1 + \cos\theta_2)$$

(a) Find the Canonical momenta p_1 and p_2. [4 marks]

(b) Show that the Lagrangian can be written in a convenient form as

$$L = \frac{1}{2}(p_1\dot{\theta}_1 + p_2\dot{\theta}_2) + mgb(2\cos\theta_1 + \cos\theta_2)$$

[8 marks]

(c) Prove that the Hamiltonian is then

$$H = \frac{1}{4mb^2}\frac{p_1^2 + 2p_2^2 - 2p_1p_2\cos(\theta_1 - \theta_2)}{1 - \frac{1}{2}\cos^2(\theta_1 - \theta_2)} - mgb(2\cos\theta_1 + \cos\theta_2)$$

[8 marks]

QUESTION 3

3. The Lagrangian for a system is given by

$$L = ml^2\left[\frac{\dot{\theta}^2}{2} + \sin^2\theta\left(\frac{\dot{\phi}^2}{2} - \omega\dot{\phi}\right)\right] - mgl(1 - \cos\theta)$$

where m, l and g are constants.

(a) Find the Hamiltonian. [12 marks]

(b) Write the equations of motion for p_θ, θ, p_ϕ, ϕ [8 marks]

QUESTION 4

4. (a) If the Hamiltonian

$$H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L$$

is expressed as a function of the generalised coordinates q_α and the momenta p_α ONLY and DOES NOT contain the time t explicitly, prove that

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

[8 marks]

(b) Given that the transformation equations for a mechanical system are given by $\mathbf{r}_\nu = \mathbf{r}_\nu(q_1, q_2, \ldots, q_n)$, where q_α are generalized coordinates. Prove that,

(i)

$$\frac{\partial \dot{\mathbf{r}}_\nu}{\partial \dot{q}_\alpha} = \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}$$

(ii)

$$\sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} = 2T$$

where T is the kinetic energy. [6, 6 marks]

QUESTION 5

5. (a) Consider the following three functions L_1, L_2 and L_3 and the Hamiltonian H for a system with three degrees of freedom:

$$\begin{aligned} L_1 &= p_2q_3 - p_3q_2 \\ L_2 &= p_3q_1 - p_1q_3 \\ L_3 &= p_1q_2 - p_2q_1 \end{aligned}$$

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + \sqrt{q_1^2 + q_2^2 + q_3^2}$$

Show that

i. $[L_1, H] = 0$ [7 marks]

ii. $[L_1, L_2] = -L_3$ [7 marks]

(b) By any method you choose, show that the following transformation is canonical

$$Q = \ln\left(\frac{\sin p}{q}\right) \quad , \qquad P = q\cot p$$

[6 marks]

QUESTION 6

6. Use the Beltrami identity ($F - y'\frac{\partial F}{\partial y'}$ = Constant) to show that the extremum for the integral

$$I = \int_0^a \sqrt{\frac{1+y'^2}{2y}}dx$$

satisfies the differential equation

$$y' = \sqrt{\frac{2c-y}{y}}.$$

By making the substitution $y = 2c\sin^2\theta$, show that the solution of the differential equation is $x = c(2\theta - \sin 2\theta)$ [20 marks]

QUESTION 7

7. Find the curve that minimizes/maximizes the following functions

(a)

$$\int_0^{\frac{\pi}{2}} (y' - \cos x)^2 \, dx$$

$y(0) = 1, \quad y\left(\frac{\pi}{2}\right) = 0$ [6 marks]

(b) Find the extremal curve of

$$I = \int_0^{\frac{\pi}{4}} (y_1^2 + y_1' y_2' + (y_2')^2) dx$$

subject to the boundary conditions $y_1(0) = 1$ and $y_1\left(\frac{\pi}{4}\right) = 2$ and $y_2(0) = \frac{3}{2}$ and $y_2\left(\frac{\pi}{4}\right)$ is not given. [7 Marks]

(c)

$$\int_{x=1}^{e} (4y + xy'^2) dx$$

if $y(e) = 0$, and $y(1)$ is not prescribed. [7 marks]