UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2007

BSc. / BEd. / B.A.S.S. III

TITLE OF PAPER	:	DYNAMICS II
COURSE NUMBER	:	M 355
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. Use the following definition

$$[F, G] = \sum_{\alpha} \left(\frac{\partial F}{\partial q_\alpha} \frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha} \frac{\partial G}{\partial q_\alpha} \right)$$

of a Poisson bracket between two physical quantities $F(q_\alpha, p_\alpha, t)$ and $G(q_\alpha, p_\alpha, t)$ to prove the following properties.

(a) $[u, v] = -[v, u]$ [3 marks]

(b) $[u, u] = 0$ [2 marks]

(c) $[u, v + w] = [u, v] + [u, w]$ [3 marks]

(d) $[u, vw] = v[u, w] + [u, v]w$ [3 marks]

(e) $[q_\alpha, p_\beta] = \delta_{\alpha\beta}$ [3 marks]

(f) $\dot{q}_\alpha = [q_\alpha, H]$ [3 marks]

(g) $\dot{p}_\alpha = [p_\alpha, H]$ [3 marks]

where q_α are generalized coordinates, p_α are generalized momenta, H is the Hamiltonian function and $\delta_{\alpha\beta}$ is the Kronecker delta.

QUESTION 2

2. Consider the system of massless pulleys connected by a light inextensible string of length l as shown in Figure 1 Taking q_1 and q_2 to be the generalized coordi-

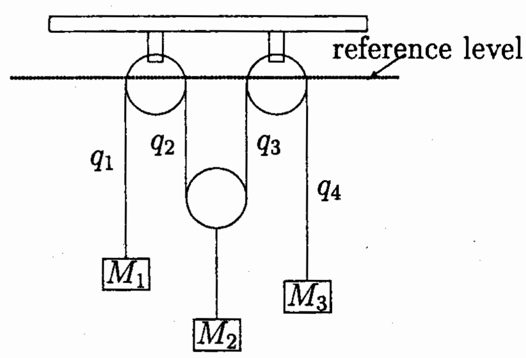

Figure 1:

nates, show that the equations of motion for the system are given by

$$(M_1 + M_3)\ddot{q}_1 + 2M_3\ddot{q}_2 = (M_1 - M_3)g$$
$$2M_3\ddot{q}_1 + (M_2 + 4M_3)\ddot{q}_2 = (M_2 - 2M_3)g$$

[20 marks]

QUESTION 3

3. (a) Find the extremal curve of

$$I = \int_0^{\frac{\pi}{2}} (y^2 - (y')^2 - 2y \sin x) dx$$

subject to the boundary conditions $y(0) = 1$ and $y\left(\frac{\pi}{2}\right) = 2$. [10 Marks]

(b) Find the extremal curve of

$$I = \int_0^{\frac{\pi}{4}} (y_1^2 + y_1' y_2' + (y_2')^2) dx$$

subject to the boundary conditions $y_1(0) = 1$ and $y_1\left(\frac{\pi}{4}\right) = 2$ and $y_2(0) = \frac{3}{2}$ and $y_2\left(\frac{\pi}{4}\right)$ is not given. [10 Marks]

QUESTION 4

4. (a) Given the following Lagrangian function

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}\kappa(x_1^2 + x_2^2) - \frac{1}{2}\kappa(x_2 - x_1)^2$$

for a certain mechanical system.

i. Find the corresponding Hamiltonian function. [6 marks]

ii. Using Hamilton's equations obtain the equations of motion for the system. [6 marks]

(b) The Hamiltonian for a system is given by

$$H = \frac{1}{a}p^a$$

where a is a constant. Given that p is a generalized momentum conjugate to the generalized coordinate q, prove that the Lagrangian for the system is given by

$$L = \left(\frac{a-1}{a}\right) \dot{q}^{\frac{a}{a-1}}$$

[8 marks]

QUESTION 5

5. (a) Given that

$$A_1 = \frac{1}{4}(x^2 + p_x^2 - y^2 - p_y^2), \quad A_2 = \frac{1}{2}(xy + p_x p_y)$$
$$A_3 = \frac{1}{2}(xp_y - yp_x), \quad A_4 = x^2 + y^2 + p_x^2 + p_y^2$$

Evaluate the following Poisson brackets

(i) $[A_1, A_2]$ [3 marks]

(ii) $[A_2, A_3]$ [3 marks]

(iii) $[A_1, A_4]$ [3 marks]

(b) Using Poisson brackets, show directly that the transformation

$$Q = \ln\left(\frac{\sin p}{q}\right), \quad P = q \cot p$$

is canonical. [5 marks]

(c) For what values of the constant parameters α and β are the following transformations canonical

(i) $Q = q^\alpha \cos \beta p$, $P = q^\alpha \sin \beta p$ [3 marks]

(ii) $Q = q^\alpha e^{\beta p}$, $P = q^\alpha e^{-\beta p}$ [3 marks]

QUESTION 6

6. (a) If the Hamiltonian H is defined by the relation $H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L$ where H is the Hamiltonian and L is the Lagrangian, show that p_α and q_α are related by the equations

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \quad \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \quad \text{and} \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

[6 marks]

(b) The Lagrangian function of a system is given by

$$L = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}k(r-b)^2$$

(i) Determine the cyclic (ignorable) coordinates. [2 marks]

(ii) Derive the Hamiltonian for the the system. [5 marks]

(iii) Using the Hamiltonian formulation show that the equation of motion corresponding to r is

$$\mu(\ddot{r} - r\dot{\theta}^2) + k(r-b) = 0$$

[7 marks]

QUESTION 7

7. (a) Suppose that the kinetic energy T does NOT contain the time t explicitly and that the potential V depends on q_α but does **NOT** depend on $\dot{q}_\alpha$. Prove that

$$\sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} = 2T$$

[6 marks]

(b) Consider a system whose dynamic behaviour is defined by the Hamiltonian $H = H(q_\alpha, p_\alpha, t)$.

(i) Prove that the equation of motion for a dynamic variable $f(q_\alpha, p_\alpha, t)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H]$$

[7 marks]

(ii) Use the above equation (i) to show that if the Hamiltonian **does not** depend on time t, then it is conserved, i.e H is a constant of motion. [7 marks]