UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2007

BSc. / BEd. / B.A.S.S. IV

TITLE OF PAPER	:	NUMERICAL ANALYSIS II
COURSE NUMBER	:	M 411
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Use the fourth-order Runge-Kutta method with step size $h = 0.1$ to approximate $y(0.2)$ and $y'(0.2)$ for the following initial value problem

$$\begin{aligned} y'' - 2y' + 2y &= e^{2x} \sin x \quad 0 \le x \le 1 \\ y(0) &= -0.4\ , \quad y'(0) = -0.6 \end{aligned}$$

[10 marks]

(b) Use the Newton's interpolating formula

$$f(x,y) \approx f_0 + \frac{(x-x_0)}{h}\Delta f_0 + \frac{(x-x_0)(x-x_1)}{2h^2}\Delta^2 f_0 + \ldots$$

to derive the Adams 3-Step formula for integrating over the interval $[k, k+1]$ assuming that information at the preceding points x_{k-2}, x_{k-1} and x_k is known. [10 marks]

QUESTION 2

2. (a) Use Taylor series method with terms through t^4 to approximate $x(0.1)$ and $y(0.1)$ as solutions of the following system of ordinary differential equations.

$$\begin{aligned} \frac{dx}{dt} &= y + tx\ , \qquad x(0) = 1 \\ \frac{dy}{dt} &= x + ty\ , \qquad y(0) = -1 \end{aligned}$$

[10 marks]

(b) Use Euler's method with $h = 0.1$ to approximate the value of $y(0.4)$ and $y'(0.4)$ in the following differential equation

$$y'' + \sin y = 0 \quad \text{with} \quad y(0) = 0\ , \quad y'(0) = 1$$

[10 marks]

QUESTION 3

3. (a) Consider the following initial value problem

$$\frac{dy}{dx} = -2x - y \qquad \text{with} \quad y(0) = -1$$

i. Find the exact solution of the differential equation at $x = 0.2$ [3 marks]

ii. Use the **modified Euler's** method with $h = 0.1$ to approximate $y(0.2)$. [7 marks]

(b) The Adams Fourth-Order Formula is given by the formula

$$y_{k+1} = y_k + \frac{h}{24}\left(55f_k - 59f_{k-1} + 37f_{k-2} - 9f_{k-3}\right)$$

Given that $y(0.3) = -0.82245$ for the differential equation

$$\frac{dy}{dx} = -2x - y \qquad \text{with} \quad y(0) = -1$$

use the above formula with $h = 0.1$ to compute $y(0.4)$ and compare your result with the exact solution at $x = 0.4$ [10 marks]

QUESTION 4

4. (a) The function u satisfies the equation

$$u_{xx} = u_{tt},$$

with boundary conditions of $u = 0$ at $x = 0$ and $u = 0$ at $x = 1$, and with initial conditions

$$u = \sin(\pi x)\ , \quad u_t = 0, \qquad \text{for} \quad 0 \leq x \leq 1$$

Write down the corresponding finite difference problem based on the central difference approximation of the derivatives, stating the [10 marks] boundary conditions in terms of the mesh points.

(b) Consider the Laplace equation over a unit square region

$\{(x, y) : 0 < x < 1, 0 \leq y \leq 1\}$:

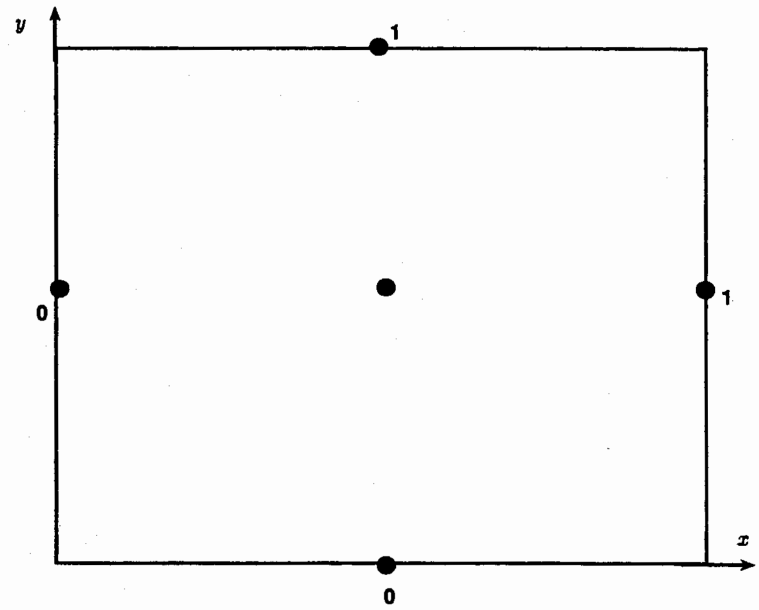
$$u_{xx} + u_{yy} = 0$$

$$u(x,0) = 0\ , \quad u(x,1) = 1; \qquad 0 \leq x \leq 1;$$

$$u(0,y) = 0, \quad u(1,y) = 0; \qquad 0 \leq y \leq 1$$

i. Using $h = k = 1/4$, write down the corresponding difference problem based on the five-point formula, stating the boundary conditions in terms of the mesh points. [6 marks]

ii. Determine the system of equations to be used to solve the problem and write them in matrix form. [4 marks]



QUESTION 5

5. (a) Consider a finite difference solution of the Poisson equation

$$u_{xx} + u_{yy} = x + y$$

on the unite square using the boundary conditions and mesh points shown on Figure 1 above. Using the second order centered difference scheme compute the approximate value of the solution at the centre of the square. [10 marks]

(b) Use the 4th-order Runge-Kutta method to solve the following initial value problem with $h = 0.1$ to estimate the given $y(0.1)$ and $z(0.1)$; [10 marks]

$$\begin{aligned} y' &= z + 1 \qquad & y(0) = 1 \\ z' &= y - x \qquad & z(0) = 1 \end{aligned}$$

QUESTION 6

6. Consider the parabolic differential equation

$$\frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 \leq x \leq 1, \quad t > 0$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0$$

$$u(x,0) = \cos 2\pi x, \quad 0 \leq x \leq 1$$

If an $O(k^2 + h^2)$ numerical method is constructed using the central difference quotient to approximate u_t and the usual difference quotient to approximate u_{xx},

(a) Write down the finite difference scheme for the problem. [10 marks]

(b) Show that the method has the matrix form

$$\mathbf{u}^{(j+1)} = \mathbf{u}^{(j-1)} + A\mathbf{u}^{(j)} \quad \text{for each} \quad j = 0, 1, 2, \ldots$$

where $\mathbf{u}^{(j)} = (u_{1,j}, u_{2,j}, \ldots, u_{m-1,j})^T$ and A is a tridiagonal matrix. [10 marks]

QUESTION 7

7. (a) Consider the differential equation

$$\frac{dy}{dx} = x + y + xy \quad \text{with} \quad y(0) = 1$$

i. Use the Taylor series method with terms through to x^3 to approximate $y(0.1)$. [8 marks]

ii. Use the Fourth Order Runge-Kutta method to approximate $y(0.1)$ using a step size of $h = 0.1$. [8 marks]

(b) Write the following Ordinary Differential equation

$$y''' + 3y''y + 6(y')^2 + 2y = 3x$$

with $y(0) = 1$, $y'(0) = 2$, $y''(0) = 3$ as a system of first order differential equations. $\left[y' = \frac{dy}{dx}\right]$ [4 marks]