UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2007

BSc. / BEd. / B.A.S.S. IV

TITLE OF PAPER	:	NUMERICAL ANALYSIS II
COURSE NUMBER	:	M 411
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. item Consider the differential equation

$$\frac{dy}{dx} = x + y + xy \quad \text{with} \quad y(0) = 1$$

   (a) Use the Taylor series method with terms through to x^4 to approximate $y(0.1)$. [10 marks]

   (b) Use the Fourth Order Runge-Kutta method to approximate $y(0.1)$ using a step size of $h = 0.1$. [10 marks]

QUESTION 2

2. (a) Write the following Ordinary Differential equation

$$y''' + 3y''y + 6(y')^2 + 2y = 3x$$

   with $y(0) = 1$, $y'(0) = 2$, $y''(0) = 3$ as a system of first order differential equations. $\left[y' = \dfrac{dy}{dx}\right]$ [8 marks]

   (b) Use Taylor series method with terms through t^4 to approximate $x(0.1)$ and $y(0.1)$ as solutions of the following system of ordinary differential equations. [12 marks]

$$\begin{aligned} \frac{dx}{dt} &= xy + t\,, \qquad x(0) = 1 \\ \frac{dy}{dt} &= yt + x\,, \qquad y(0) = -1 \end{aligned}$$

QUESTION 3

3. (a) Use the Newton's interpolating formula

$$f(x,y) \approx f_0 + \frac{(x-x_0)}{h}\Delta f_0 + \frac{(x-x_0)(x-x_1)}{2h^2}\Delta^2 f_0 + \dots$$

to derive the following Adams 3-Step formula.

$$y_{k+1} = y_k + h\left(\frac{23}{12}f_k - \frac{4}{3}f_{k-1} + \frac{5}{12}f_{k-2}\right)$$

for integrating over the interval $[k, k+1]$ assuming that information at the preceding points x_{k-2}, x_{k-1} and x_k is known. [10 marks]

(b) Use the 3-Step Adams formula with $h = 0.2$ to approximate $y(0.6)$ if

$$\frac{dy}{dx} = x - y \qquad \text{if} \quad y(0) = 1, \quad y(0.2) = 0.837467, \quad y(0.4) = 0.740649$$

[5 marks]

(c) Find the exact solution of the differential equation and compare it with the solution from (b) to determine the error of the 3-Step formula. [5 marks]

QUESTION 4

4. (a) The function u satisfies the equation

$$u_{xx} = u_{tt},$$

with boundary conditions of $u = 0$ at $x = 0$ and $u = 0$ at $x = 1$, and with initial conditions

$$u = \sin(\pi x)\ , \quad u_t = 0, \qquad \text{for} \quad 0 \le x \le 1$$

Write down the corresponding finite difference problem based on the central difference approximation of the derivatives, stating the boundary conditions in terms of the mesh points. [10 marks]

(b) Consider the Laplace equation over a unit square region

$\{(x, y) : \ 0 < x < 1, \ 0 \leq y \leq 1\}$:

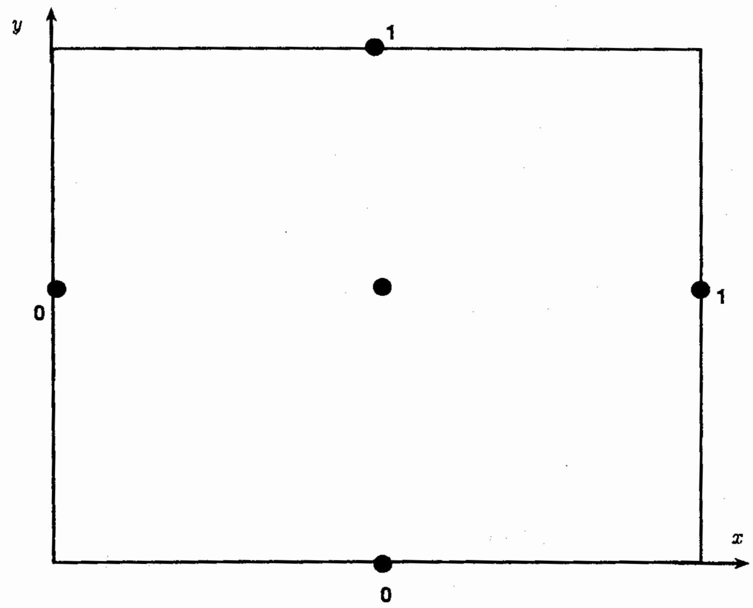
$$u_{xx} + u_{yy} = 0$$

$$u(x, 0) = 0 \ , \quad u(x, 1) = 1; \qquad 0 \leq x \leq 1;$$

$$u(0, y) = 0, \quad u(1, y) = 0; \qquad 0 \leq y \leq 1$$

i. Using $h = k = 1/4$, write down the corresponding difference problem based on the five-point formula, stating the boundary conditions in terms of the mesh points. [6 marks]

ii. Determine the system of equations to be used to solve the problem and write them in matrix form. [4 marks]



QUESTION 5

5. (a) Consider a finite difference solution of the Poisson equation

$$u_{xx} + u_{yy} = x + y$$

on the unite square using the boundary conditions and mesh points shown on Figure 1 above. Using the second order centered difference scheme compute the approximate value of the solution at the centre of the square. [10 marks]

(b) Use the 4th-order Runge-Kutta method to solve the following initial value problem with $h = 0.1$ to estimate the given $y(0.1)$ and $z(0.1)$; [10 marks]

$$\begin{aligned} y' &= z + 1 \qquad & y(0) = 1 \\ z' &= y - x \qquad & z(0) = 1 \end{aligned}$$

QUESTION 6

6. Consider the following Poisson equation over the square region $\{(x, y) : 0 < x < 1, 0 \leq y \leq 1\}$:

$$u_{xx} + u_{yy} = x$$

$$u(x, 0) = u(x, 1) = x^2; \qquad 0 \leq x \leq 1;$$

$$u(0, y) = 0, \quad u(1, y) = y^2; \qquad 0 \leq y \leq 1$$

(a) Using $h = k = 1/3$, write down the corresponding difference problem based on the five-point formula, stating the boundary conditions in terms of the mesh points. [10 marks]

(b) Determine the system of equations to be used to solve the problem and write them in matrix form. [10 marks]

QUESTION 7

7. Consider the parabolic differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le 1, \qquad t > 0$$

$$u(0,t) = 4, \qquad u(1,t) = 4, \qquad t > 0$$

$$u(x,0) = 0, \qquad 0 \le x \le 1$$

If an $O(k + h^2)$ numerical method is constructed using the implicit backward difference quotient to approximate u_t and the usual central difference quotient to approximate u_{xx}, show that the resulting difference problem can be written in matrix form as [20 marks]

$$\mathbf{u_j} = \mathbf{A}\mathbf{u_{j+1}} + \mathbf{B_j}$$