University of Swaziland

Final Examination, December 2006

BSc IV, Bass IV, BEd IV

Title of Paper : Partial Differential Equations
Course Number : M415
Time Allowed : Three (3) hours
Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) For each of the following PDEs, state the order, whether it is homogeneous or not, linear or non-linear. [4 marks]

(i) $x^{\frac{3}{2}}u_{xx} + y^2u_{xy} - u = 4\sin(x+y)$

(ii) $u_x^2 - uu_y = u$

(iii) $u_{xxxx} + 4xu = 0$

(iv) $x^2u_{yy} - yu_{xxx} = \tan u$

(b) Find a particular solution of $u_{xy} = 4xy$ that satisfies the conditions $u = 0$ and $u_x = 1$ along $x - y = 0$. [10 marks]

(c) Consider

$$u(x,y) = e^{-x}f(2x-3y), \tag{1}$$

in which f is an arbitrary function. There exists a unique PDE (call it uniq-PDE) for which (1) is a general solution.

(i) State the order of uniq-PDE and whether is expected to be linear or not. Give a reason for your answer in each case. [2 marks]

(ii) Find uniq-PDE. [4 marks]

Question 2

(a) The two-dimensional Laplace equation in Cartesian coordinates is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{2}$$

(i) Show that the substitution $x = \rho\cos\varphi,\ y = \rho\sin\varphi$ transforms (2) into

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} = 0.$$ [12 marks]

(ii) Compute the Jacobian and hence comment about when the transformation will break down. [2 marks]

(b) Consider the system

$$\begin{aligned} u_t &= u_{xx}, \quad 0 \le x \le 1, \ t \ge 0, \\ u(x,0) &= x, \quad 0 \le x \le 1, \\ u(0,t) &= u(1,t) = 0, \ t > 0. \end{aligned}$$

Give a full description of the physical process specified by this system. Your description should include the type of process, a sketch of the initial profile of u, and profiles at two subsequent times. [6 marks]
NB: *You are not asked to solve the system in this question.*

Question 3

Consider the PDE

$$yu_{xx} + (x+y)u_{xy} + xu_{yy} = 4, \quad y \ne x.$$

(a) Classify it as hyperbolic, parabolic or elliptic. [4 marks]
(b) Reduce the equation into its canonical form and hence find its general solution. [16 marks]

Question 4

Consider the Cauchy problem for the wave equation

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad -\infty < x < \infty, \ t \ge 0, \\ u(x,0) &= f(x), \quad -\infty < x < \infty, \\ u_t(x,0) &= g(x), \quad -\infty < x < \infty. \end{aligned}$$

Derive the d'Alembert's solution

$$u(x,t) = \tfrac{1}{2}\left\{f(x+ct) + g(x-ct)\right\} + \tfrac{1}{2c}\int_{x-ct}^{x+ct} g(\alpha)\alpha. \qquad [20 \text{ marks}]$$

Question 5

Find the solution of the steady-state problem [20 marks]

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad 0 \le x, y \le 1, \\ u(x,0) &= x(x-1), \ 0 \le x \le 1, \\ u(x,1) &= 0, \quad 0 \le x \le 1, \\ u(0,y) &= u(1,y) = 0, \ 0 \le y \le 1. \end{aligned}$$

Question 6

(a) Find the particular solution of the PDE

$$yu_x + x^2 u_y = xy$$

which contains the curve $u = \frac{1}{2}x^2$ along $3y^2 - 4$. [8 marks]

(b) Consider the PDE

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0.$$

Show that

$$u(x, y) = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right),$$

where f and g are arbitrary functions, is a general solution. [12 marks]

Question 7

The initial temperature distribution in a thin insulated circular unit disc is given by $f(r) = 1 - r^2$. If the disc is allowed to cool down with its circular edge kept at temperature zero, the subsequent temperature distribution develops according to the system

$$\begin{aligned} u_t &= u_{rr} + \tfrac{1}{r}u_r, \quad 0 \le r \le 1,\ t \ge 1, \\ u(r, 0) &= 1 - r^2,\ 0 \le r << 1, \\ u(1, t) &= 0, \qquad t \ge 0. \end{aligned}$$

Solve this system to find the subsequent temperature distribution of the disc. [20 marks]

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