UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2007

BSc./ B.Ed./ BASS IV

TITLE OF PAPER: ABSTRACT ALGEBRA II

COURSE NUMBER: M423

TIME ALLOWED: THREE HOURS

INSTRUCTIONS:

1. This paper consists of SEVEN questions on FOUR pages.
2. Answer any FIVE questions.
3. Calculators may be used.

SPECIAL REQUIREMENTS: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

## Question 1

(a) Let $f$ be a polynomial over $\mathbb{Z}$, which is irreducible over $\mathbb{Z}$. Show that $f$ considered as a polynomial over $\mathbb{Q}$ is also irreducible

**[10 marks]**

(b) Classify each of the given $\alpha \in \mathbb{C}$ as algebraic or transcedental over the given field $F$. If $\alpha$ is algebraic over $F$ find $deg(\alpha, F)$

(i) $\alpha = 1 + i, F = \mathbb{Q}$
(ii) $\alpha = \sqrt{\pi}, F = \mathbb{Q}[\pi]$
(iii) $\alpha = \pi^2, F = \mathbb{Q}$
(iv) $\alpha = \pi^2, F = \mathbb{Q}(\pi)$
(v) $\alpha = \pi^2, F = \mathbb{Q}(\pi^3)$

**[10 marks]**

## Question 2

(a) Prove that if $D$ is an integral domain, then $D[x]$ is also an integral domain.

**[8 marks]**

(b) Decide the irreducibility or otherwise of
(i) $x^3 - 7x^2 + 3x + 3 \in \mathbb{Q}[x]$

**[6 marks]**

(ii) $2x^{10} - 25x^3 + 10x^2 - 30 \in \mathbb{Q}[x]$

**[6 marks]**

## Question 3

(a) In a ring $\mathbb{Z}_n$ show that the
(i) divisors of zero are those elements that are NOT relatively prime to $n$.
**[5 marks]**
(ii) elements that are relatively prime can't be zero divisors
**[5 marks]**

(b) (i) Give an example of a ring $R$ with unity 1 that has a subring $\mathbb{R}^1$ with unity $1^1$; where $1 \neq 1^1$.
(ii) Describe all units in the ring $\mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}$
**[4 marks]**

## Question 4

(a) (i) Show that the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$ is NOT a field
**[5 marks]**
(ii) Find a polynomial of degree $> 0$ in $\mathbb{Z}_4[x]$ that is a unit
**[5 marks]**

(b) (i) Show that $(a + b)(a - b) = a^2 - b^2$ for all $a$ and $b$ in a ring $R$, if and only if $R$ is commutative **[6 marks]**
(ii) Show that the rings $2\mathbb{Z}$ and $3\mathbb{Z}$ are NOT isomorphic
**[4 marks]**

## Question 5

(a) (i) Define an ideal $N$ of a ring $R$.
(ii) Find all ideals $N$ of $\mathbb{Z}_{12}$ and all maximal ideals of $\mathbb{Z}_{18}$

**[8 marks]**

(b) (i) Prove that every finite integral domain is a field.

**[6 marks]**

(ii) Show that for a field $F$, the set of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \quad \text{for} \quad a, b \in F$$

is a right ideal but not a left ideal of $M_2(F)$.

**[6 marks]**

## Question 6

(a) Suppose $F$ is a field, $f$ is an irreducible polynomial over $F$ and $g, h$ are polynomials over $F$ such that $F$ divides $gh$. Show that either $f$ divides $g$ or $f$ divides $h$.

**[10 marks]**

(b) Let $\mathbb{Q}_\alpha : \mathbb{Z}_7[x] \to \mathbb{Z}_7$. Evaluate each of the following for the indicated evaluation homorphism.
(i) $\varphi_5[(x^3 + 2)(4x^2 + 3)(x^7 + 3x^2 + 1)]$

**[5 marks]**

(ii) $\varphi_4(3x^{106} + 5x^{99} + 2x^{53})$

**[5 marks]**

## Question 7

(a) Determine whether each of the following polynomials in $\mathbb{Z}[x]$ satisfies an Eisenstein criteria for irreducibility

(i) $8x^3 + 6x^2 - 9x + 24$

**[4 marks]**

(ii) $2x^{10} - 25x^3 + 10x^2 - 30$

**[4 marks]**

(b) Let $\alpha$ be a zero of $x^2 + 1$ in an extension field of $\mathbb{Z}_3$. Give the multiplication and addition tables for the nine elements of $\mathbb{Z}_3(\alpha)$

**[12 marks]**

******** *END OF EXAMINATION* ********