UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2007

B.Sc. / B.Ed. / B.A.S.S. IV

TITLE OF PAPER	:	Metric Spaces
COURSE NUMBER	:	M431
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Let X be a nonempty set.

(i) What is meant by saying that (X, d) is a *metric space*?

(ii) Let d be the function on $\mathbb{R}^2$ defined by

$$d(x, y) = 2|x_1 - y_1| + 3|x_2 - y_2|,$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$. Prove that $(\mathbb{R}^2, d)$ is a metric space. [10]

(b) Let (X, d) be a metric space. Define the following:

(a) the distance from $x \in X$ to a subset $A \subset X$,

(b) the diameter of $A \subset X$,

(c) the distance between two subsets, A and B, of X,

(d) a bounded subset $A \subset X$,

(e) a bounded mapping g from a nonempty set Y to X. [10]

QUESTION 2

(a) Let (X, d) be a metric space and (x_n) be a sequence in X. What is meant by saying that (x_n) is *convergent*? [2]

(b) Decide whether or not the following sequences are convergent in the usual (Euclidean) metric on $\mathbb{R}^2$:

(i) $x_n = \left(\frac{n^2}{2n^2+1}, \frac{1}{n+1}\sin(\frac{n\pi}{2})\right)$,

(ii) $x_n = (3^{-n}, (-1)^n \exp(\frac{1}{n}))$. [8]

(c) (i) Suppose that (x_n) converges to x in $C[a,b]$ in the uniform metric. Explain what is meant by *pointwise convergence*. Show that (x_n) converges to x pointwise.

(ii) Let x_n in $C[0,1]$ be defined by

$$x_n(t) = \begin{cases} nt & \text{if } 0 \le t \le \frac{1}{n}, \\ 1 & \text{if } \frac{1}{n} \le t \le 1. \end{cases}$$

Sketch the graph of $x_n(t)$ and show that (x_n) converges pointwise to the function

$$x(t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } 0 < t \le 1. \end{cases}$$

Deduce that (x_n) is not convergent in $C[0,1]$. [10]

QUESTION 3

(a) Define what is meant by:

(i) a *Cauchy sequence* in a metric space,

(ii) a *complete metric space*. [3]

(b) Which of the following spaces X is complete and which is incomplete in the usual (Euclidean) metric? Give reasons.

(i) $X = \mathbb{Q}$,

(ii) $X = \{\frac{1}{n} : n \in \mathbb{N}\}$. [4]

(c) Let (X,d) be a metric space with the metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 3 & \text{if } x \neq y \end{cases}$$

Show that any Cauchy sequence in X is eventually constant, and deduce that (X,d) is complete. [5]

(d) (i) Explain what is meant by a *contraction* of a metric space. Show that if $f : [a,b] \longrightarrow [a,b]$ is differentiable, then f is a contraction if and only if there is number $r < 1$ such that $|Df(x)| \leq r$ for every $x \in (a,b)$.

(ii) State without proof the *Contraction Mapping Theorem.*

(iii) Show that the mapping $f : [-1,1] \longrightarrow [-1,1]$ defined by $f(x) = \frac{1}{14}(3x^3 - 2x^2 + 9)$ is a contraction, and deduce that there is unique solution to the equation $3x^3 - 2x^2 - 14x + 9 = 0$ in the interval $[-1,1]$. [8]

QUESTION 4

(a) Let (X,d) be a metric space and let $S \subseteq X$. What is meant by saying that A is open? Show that if $(A_i)_{i\in I}$ is any collection of open sets, then the union $\bigcup_{i\in I} A_i$ is also open. [6]

(b) What is meant by an *open ball* $B(a,r)$ in a metric space (X,d)? Show that an open ball is open. By drawing a diagram, or otherwise, describe the open ball $B(a,3)$ in $\mathbb{R}^2$, where $a = (3,4)$

(i) with the usual metric,

(ii) with the max metric. [8]

(c) Show that $\emptyset$ and X are open, where (X,d) is a metric space. [6]

QUESTION 5

(a) Let $f : X \longrightarrow Y$, where X and Y are metric spaces. Give the definition of continuity of f in terms of convergence of sequences. Show that if f is continuous, then the following are true:

(i) if A is a closed subset of Y, then $f^{-1}(A)$ is a closed subset of X.

(ii) if $Y = X$, then $f^2 : X \longrightarrow X$ is also continuous (where $f^2(x) = f(f(x))$).[10]

(b) Suppose that $f, g : X \longrightarrow \mathbb{R}$ are both continuous. Show that the function $h : X \longrightarrow \mathbb{R}$ defined by

$$h(x) = 6f(x) - 5g(x)$$

is continuous. [4]

(c) Let f be the function $f : C[0,1] \longrightarrow \mathbb{R}$ defined for $x \in C[0,1]$ by $f(x) = x(0)$. Show that f is not continuous with respect to the L_1 metric on $C[0,1]$ (and the usual metric on $\mathbb{R}$) by considering the functions $x_n(t)$ given by

$$x_n(t) = \begin{cases} (n-1)t & \text{if } 0 \leq t \leq \frac{1}{n} \\ 1-t & \text{if } \frac{1}{n} \leq t \leq 1 \end{cases}$$

(**Hint** Sketch the functions $x_n(t)$ and consider their limit in the L_1 metric). [6]

QUESTION 6

(a) Let X be a metric space and $A \subseteq X$. What is meant by saying that A is *compact*? [2]

(b) Assuming that a closed bounded subset of $\mathbb{R}$ is compact, show that the same is true for $\mathbb{R}^2$. [7]

(c) Show that in any metric space, a closed subset of a compact set is compact. [5]

(d) Which of the following sets is compact? Give reasons.

(i) $\{(x,y) : 0 \leq x < y \leq 1\}$ in $\mathbb{R}^2$,

(ii) $\{1, \frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^n}, \ldots\}$ in $\mathbb{R}$. [6]

QUESTION 7

(a) Let X be a set and let d_1 and d_2 be metrics on X. What is meant by saying that the metrics d_1 and d_2 are *equivalent*? [3]

(b) Suppose that there are positive constants k and K

$$kd_1(x, y) \leq d_2(x, y) \leq Kd_1(x, y)$$

for all $x, y \in X$. Show that d_1 and d_2 are equivalent. [5]

(c) Show that on $\mathbb{R}^2$, the Euclidean metric and the Chicago metric are equivalent. [4]

(d) Suppose that d_1 and d_2 are equivalent metrics on a set X. Show that if $B_1(a, \varepsilon)$ is any open ball in the d_1 metric, then there is an open ball $B_2(a, \delta)$ in the d_2 metric with $B_2(a, \delta) \subseteq B_1(a, \varepsilon)$. Deduce that if $x_n \to x$ in the d_2 metric, then it is convergent in the d_1 metric. [8]

END OF EXAMINATION