UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2007

BSc. / BEd. / B.A.S.S. IV

TITLE OF PAPER	:	FLUID DYNAMICS
COURSE NUMBER	:	M 455
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) The velocity field for a fluid motion is given by

$$\mathbf{q} = Ax^3y^2z\mathbf{i} + (z^3 + Bx^4yz)\mathbf{j} + (3yz^2 - x^4y^2)\mathbf{k}$$

find the values of the constants A and B if the flow is irrotational and derive the velocity potential ϕ for this flow. [8 marks]

(b) Given that the velocity field for a fluid flow is given by

$$\mathbf{q} = x^2t\mathbf{i} + 2xyt\mathbf{j} + 2yzt\mathbf{k}$$

find the *vorticity* and *rotation* at $(2, -1, 3)$ when $t = 2$. [6 marks]

(c) Show that the stream function ψ corresponding to the complex potential $w = z^2 + z + \dfrac{a^4}{z^2} + \dfrac{a^2}{z}$ is given by

$$\psi = 2xy + y - \frac{2a^4xy}{(x^2+y^2)^2} - \frac{a^2y}{x^2+y^2}$$

[6 marks]

QUESTION 2

2. A container is formed by rotating the curve $z = \frac{x^2}{a}$ about the vertical z-axis. The container is filled with liquid to a depth h. At time $t = 0$ a small hole of radius $\frac{a}{n}$ (n being very large) is opened at the bottom so that the liquid drains out. Let $z(t)$ be the depth of the remaining liquid at time t. Show that

$$\frac{dz}{dt} = -a\sqrt{\frac{2gz}{n^4z^2 - a^2}}$$

[12 marks]

and hence show that, *approximately*,

$$z(t) = \left(h^{\frac{3}{2}} - \frac{3at\sqrt{g}}{\sqrt{2}n^2}\right)^{2/3}$$

[8 marks]

QUESTION 3

3. A viscous liquid occupies the space between two coaxial, infinitely-long cylinders. The inner cylinder has radius a and is fixed, while the outer cylinder has radius b and is rotating with constant angular velocity Ω. Let (r, θ, z) be cylindrical polar coordinates with z-axis coinciding with the cylinders' axis such that the outer cylinder is rotating in the direction of increasing θ. Assuming that the velocity of the liquid has the form $\mathbf{q} = u(r)\hat{\theta}$ (where $u(r)$ means that u is a function of r only and $\hat{\theta}$ is a unit vector in the θ direction) and that body forces are negligible,use the Navier-Stokes equations in the form

$$\nabla(\frac{1}{2}\mathbf{q}^2) - \mathbf{q} \times (\nabla \times \mathbf{q}) = -\frac{1}{\rho}\nabla p - \nu\nabla \times (\nabla \times \mathbf{q})$$

to show that $u = A(r - a^2/r)$, where $A = \Omega(1 - a^2/b^2)$. [20 marks]

QUESTION 4

4. (a) The velocity potential for a steady incompressible, irrotational flow with circulation around a fixed cylinder of radius a is given in cylindrical polar coordinates by

$$\phi = Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta + \frac{k\theta}{2\pi}$$

where U is the uniform speed at infinity. Find the corresponding

(i) velocity field $\mathbf{q}$. [4 marks]

(ii) stream function ψ. [6 marks]

(b) In the two $z = x + iy$ plane, a line vortex of strength $m > 0$, is placed at $z = c$ and another, of strength $-m$, at $z = -c$, where c is a real positive number. Both vortices are held fixed at these locations. Write down the complex potential w for this flow and show that the stream function ψ and the velocity potential ϕ are given by

$$\psi = \frac{m}{4\pi}\log\frac{(x-c)^2 + y^2}{(x+c)^2 + y^2} \quad \text{and} \quad \phi = -\frac{m}{2\pi}\tan^{-1}\frac{2cy}{x^2 + y^2 - c^2}$$

hint: You may set $z - c = r_1 e^{i\theta_1}$ and $z + c = r_2 e^{i\theta_2}$, and use $\tan^{-1} A + \tan^{-1} B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$ [10 marks]

QUESTION 5

5. (a) Show that the complex velocity potential corresponding to the velocity field

$$\mathbf{q} = 3(y^2 - x^2)\mathbf{i} + 6xy\mathbf{j} \text{ is } w(z) = z^3$$

[6 marks]

(b) A fluid has the complex velocity potential

$$w(z) = i + z^2, \quad z = x + iy$$

Find the stream function and the velocity potential for this flow [4 marks]

(c) Consider the viscous flow of fluid confined between two parallel flat plates of infinite extent in the xz plane. The distance between the plates is h with the lower plate fixed at $y = -h/2$ and the upper plate is fixed at $y = h/2$. If the velocity field for the flow is of the form

$$\mathbf{q} = (u(y), 0, 0)$$

use Navier-Stokes equations in the form

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla)q = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{q}$$

to show that the velocity profile for this flow is

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}\left\{y^2 - \frac{h^2}{4}\right\}$$

[10 marks]

QUESTION 6

6. (a) An incompressible fluid flows steadily past a sphere of radius a located at the origin. The velocity at large distances away from the sphere is $U\mathbf{i}$, where $\mathbf{i}$ is a unit vector in the x-direction. It is assumed that the motion is irrotational and that there are no body forces. Given that the velocity potential for the flow is given by

$$\phi = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta$$

in the usual spherical coordinates (r, θ, ψ), with $\theta = 0$, measured from the x-axis.

i. Find the velocity components of the flow. [5 marks]

ii. Show that the pressure is given by

$$p = p_\infty + \frac{1}{2}\rho U^2\left(1 - \frac{9}{4}\sin^2\theta\right)$$

where p_∞ is the pressure at large distances from the sphere. [5 marks]

(b) At a point in a steady, incompressible fluid having cylindrical coordinates (r, θ, z) the velocity components are

$$(r^2\cos\theta, -3r^2\sin\theta, 0)$$

Determine whether or not the equation of continuity is satisfied, and if so find the equations of the streamlines [10 marks]

QUESTION 7

7. Consider the boundary layer equations in the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - au \quad (2)$$

with boundary conditions

$$u = ax \quad , \quad v = -(\nu a)^{\frac{1}{2}} \quad \text{on y} = 0 \quad \text{and} \quad u = 0 \quad \text{on} \quad y = \infty$$

Using the similarity transformation $\eta = y\left(\frac{a}{\nu}\right)^{\frac{1}{2}}$ and the stream function formulation $\psi = -x(\nu a)^{\frac{1}{2}} f(\eta)$ where a is a constant and ν is the dynamic viscosity, Show that equation (2) and the boundary conditions can be transformed into

$$f''' + ff'' - (f')^2 - f' = 0$$

$$f = 1 \quad , \quad f' = 1 \quad \text{on } \eta = 0 \quad \text{and} \quad f' = 0 \quad \text{on } \eta = \infty$$

[20 marks]