UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2007/8

BSc. /B.Ed. /B.A.S.S

TITLE OF PAPER	:	LINEAR ALGEBRA
COURSE NUMBER	:	M 220
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Find conditions for λ and μ for which the following system of linear equations has

   i. a unique solution

   ii. no solution

   iii. infinitely many solutions

$$\begin{aligned} x+y+z &= 0 \\ 2x+3y+z &= 1 \\ 4x+7y+\lambda z &= \mu \end{aligned}$$

[10]

(b) Let $T:\mathbb{R}^2 \to \mathbb{R}^3$ be given by $T(x,y) = (x-2y, 2x+y, x+y)$. Find the matrix of T.

   i. with respect to the standard basis

   ii. with respect to B^1 and B where $B^1 = \{(1,-1),(0,1)\}$ and $B = \{(1,1,0),(0,1,1),(1,-1,1)\}$ [10]

QUESTION 2

2. (a) Determine whether the following sets of vectors in the vector space $P_2(x)$ are linearly dependent. For those that are linearly dependent express the last vector as a linear combination of the rest.

   i. $\{2x^2+x, x^2+3, x\}$

   ii. $\{2x^2+x+1, 3x^2+x-5, x+13\}$ [10]

(b) Let $S = \{v_1, v_2, \cdots, v_n\}$ be a set of non-zero vectors in a vector space V. Prove that S is linearly dependent if and only if one of the vectors v_i is a linear combination of the preceeding vectors in S. [10]

QUESTION 3

3. (a) Let V be the set of all ordered pairs of real numbers. Define addition and scalar multiplication as follows

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$$

and $\alpha(x_1, y_1) = (\alpha x_1 + \alpha - 1, \alpha y_1 + \alpha - 1)$. Show that V is a vector space. [8]

(b) Find the inverse A^{-1} of the matrix A using the augmented matrix $[A : I]$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

[4]

(c) Use (b) to find a finite sequence of elementary matrices $E_1 E_2, \cdots, E_k$ such that $E_k, E_{k-1} \cdots E_2 E_1 A = I$. [8]

QUESTION 4

4. (a) Show that the vector $(-3, 12, 12)$ is a linear combination of the vectors $(1, 0, 2)$, $(0, 2, 4)$ and $(-1, 3, 2)$ [8]

(b) Show that the set of vectors

$$V = \{(0, 2, 1), (1, 0, 2), (1, -1, 0)\}$$

is a basis for $\mathbb{R}^3$. [8]

(c) Determine whether the following has a non-trivial solution

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ 2x_1 + x_2 - x_3 + 2x_4 &= 0 \\ 3x_1 + 2x_2 + 2x_3 + 2x_4 &= 0 \end{aligned}$$

[4]

QUESTION 5

5. (a) Show that each eigenvector of a square matrix A is associated with only one eigenvalue. [5]

(b) Show that $A = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$ is skew-symmetric [5]

(c) Find the characteristic polynomial, eigenvalues and eigenvectors of the following matrix

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix}$$ [10]

QUESTION 6

6. Let $B = \{u_1, u_2, u_3\}$ and $B^1 = \{v_1, v_2, v_3\}$ be bases in $\mathbb{R}^3$, where $u_1 = (1,0,0)$, $u_2 = (1,1,0)$, $u_3 = (1,1,1)$, $v_1 = (0,2,1)$, $v_2 = (1,0,2)$ and $u_1 = (1,-1,0)$.

(a) Find the transition matrix from B^1 to B. [8]

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix with respect to the basis B is $\begin{pmatrix} 3 & -6 & 9 \\ 0 & 3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$. Find the matrix of T with respect to B^1. [8]

(c) If $\begin{pmatrix} 6 \\ -3 \\ 3 \end{pmatrix}$ are the coordinates relative to B, find the coordinates relative to B^1 [4]

QUESTION 7

7. (a) Verify Cayley-Hamilton theorem for the following matrix

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$$

[5]

(b) By inspection, find the inverses of the following elementary matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[5]

(c) Use Cramer's rule to solve

$$\begin{aligned} 2x_1 + 8x_2 + x_3 &= 10 \\ 2x_3 + 3x_2 - x_1 &= -2 \\ 4x_1 + 4x_2 - 5x_3 &= 4 \end{aligned}$$

[5]

(d) Show that $2x^2 + 2x + 3$ is not a linear combination of

$$x^2 + 2x + 1, \quad x^2 + 3, \quad x - 1$$

[5]