UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2007/2008

B.Sc. / B.Ed. / B.A.S.S. II

TITLE OF PAPER	:	FOUNDATIONS OF MATHEMATICS
COURSE NUMBER	:	M231
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Define the following:

(i) Axiom;

(ii) Conjecture;

(iii) Theorem;

(iv) Corollary. [4]

(b) Use the quantifiers *for all* or *there is*, or their equivalents, to make each of the following a true statement:

(i) $(x-1)^2 = x^2 - 2x + 1$;

(ii) $|x| = x$. [4]

(c) Let P be the statement "All girls are good at mathematics." Which of the following statements is the negation of P?

(i) All girls are bad at mathematics;

(ii) All girls are not good at mathematics;

(iii) Some girl is bad in mathematics;

(iv) Some girl is not good at mathematics;

(v) All children who are good at mathematics are girls;

(vi) All children who are not good at mathematics are boys;

Can you find any statement in this list that has the same meaning as statement P? [6]

(d) Find the negation of the statement "For every real number x between -1 and 1, there is a real number y between -1 and 1 such that $(x^2 + y^2) \leq 1$." [6]

QUESTION 2

(a) What do you understand by the following?

(i) Premiss;

(ii) Deductive reasoning;

(iii) Inductive reasoning;

(iv) Mathematical Proof. [4]

(b) Using the axioms given below, prove each of the theorems which follow.

Axiom 1 All mathematicians are logical.

Axiom 2 Careful people are not foolish.

Axiom 3 Discontented people are foolish.

Axiom 4 Logical people are careful.

Theorem 1 *Mathematicians are contented.*

Theorem 2 *Foolish people are not logical.*

Theorem 3 *Careless people are not mathematicians.*

[6]

(c) Consider the proposition "If z is a real number that satisfies $z^3+3z^2-9z-27 \geq 0$, then $|z| \geq 3$."

(i) Reword the proposition so that it is in the form "A implies C OR D." [2]

(ii) Prove the proposition by assuming that A and $NOT\,C$ are true. [4]

(iii) Prove the proposition by assuming that A and $NOT\,D$ are true. [4]

QUESTION 3

(a) Four intelligent frogs sit on a log; two green frogs on one side and two brown frogs on the other side, with an empty seat separating them. They decide to switch places. The only moves permitted are to jump over one frog of a different color into an empty space or to jump into an adjacent space. What is the minimum number of moves? Generalize this problem and solve it. [10]

(b) Critic Ivor Smallbrain has been thrown into prison for libeling the great film director Michael Loser. During one of his needlework classes in prison, Ivor is given a pile of pieces of leather in the shapes of regular pentagons and regular hexagons, and is told to sew some of these together into a convex polyhedron (which will then be used as a soccer ball). He is told that each vertex must lie on exactly 3 edges (that is, the corresponding plane graph is regular of degree 3). Ivor immediately exclaims, "Then I need exactly 12 pentagonal pieces."

Prove that Ivor is correct. [10]

QUESTION 4

(a) State and prove the Principle of Mathematical Induction I. [6]

(b) Find a formula for the sum

$$\frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{n(n+1)}.$$

Prove that your formula is correct. [8]

(c) Use the principle of Mathematical Induction II to prove that the sum of the internal angles in an n-sided polygon is $(n-2)\pi$. [6]

QUESTION 5

(a) Prove that the square root of any prime number is irrational. [10]

(b) Critic Ivor Smallbrain is watching the classic film $11.\overline{9}$ *Angry Men.* But he is bored and starts wondering idly exactly which rational numbers $\frac{m}{n}$ have decimal expressions ending in $0000\ldots$ (that is, ending in repeating zeros). He notices that this is the case if the denominator n is 2, 4, 5, 8, 10, or 16, and wonders if there is a simple general rule which tells us which rational numbers have this property.

Help Ivor by proving that a rational number $\frac{m}{n}$ (in its lowest terms) has a decimal expression ending in repeating zeros if and only if the denominator n is of the form $2^a 5^b$, where a and b are integers with $a, b \geq 0$. [10]

QUESTION 6

(a) Express $1.813813813\cdots$ as a fraction $\frac{m}{n}$, where $m, n \in \mathbb{Z}$ with $n \neq 0$. [5]

(b) Show that if $a_0.a_1a_2a_3\ldots$ and $b_0.b_1b_2b_3\ldots$ are two different decimal representations of the same real number, then one of them ends in $9999\cdots$ and the other in $0000\cdots$. [8]

(c) Prove that a real number is rational if and only if its decimal representation is repeating. [7]

QUESTION 7

(a) Give the definition of a countable set. [3]

(b) Show that if A and B are countable sets, then $A \cup B$ is also countable. [9]

(c) Prove that the set of all rational numbers is countable. [8]

END OF EXAMINATION