UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2007/2008

B.Sc. / B.Ed. / B.A.S.S.III

TITLE OF PAPER	:	VECTOR ANALYSIS
COURSE NUMBER	:	M312
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Find a parametrization of the cylinder

$$x^2 + (y-a)^2 = a^2 \qquad 0 \le z \le 5,$$

where a is a constant. [6]

(b) Find the distance from the plane $x + 2y + 6z = 10$ to the plane $x + 2y + 6z = 20$. [7]

(c) Find the angle between the planes $x + y = 1$ and $2x + y - 2z = 2$. [7]

QUESTION 2

(a) A curvilinear coordinate system (u, v, ϕ) is defined by

$$x = auv\cos\phi, \qquad y = auv\sin\phi, \qquad z = \tfrac{a}{2}(u^2 - v^2), \text{ where } u, v > 0,\ -\pi < \phi < \pi.$$

(i) Find the scale factors and the unit vectors.

(ii) Show that the coordinate system is orthogonal.

(iii) Find the line element and the volume element. [12]

(b) Show that $y\mathrm{d}x + x\mathrm{d}y + 4\mathrm{d}z$ is exact and evaluate the integral

$$\int_{(0,0,0)}^{(2,3,-1)} y\mathrm{d}x + x\mathrm{d}y + 4\mathrm{d}z.$$

[8]

QUESTION 3

(a) Find out which of the fields given below are conservative. For conservative fields, find a potential function.

(i) $\mathbf{F} = (z + y)\hat{\mathbf{i}} + z\hat{\mathbf{j}} + (xy\cos z)\hat{\mathbf{k}}$.

(ii) $\mathbf{F} = (y\sin z)\hat{\mathbf{i}} + (x\sin z)\hat{\mathbf{j}} + (y\cos z)\hat{\mathbf{k}}$. [12]

(b) Integrate $f(x, y, z) = 2x - 6y^2 + 2z$ over the line segment C joining the points (1,1,1) and (3,3,3). [8]

QUESTION 4

(a) Find the surface area of a hemisphere of radius a cut off by a cylinder having a diameter of a. [10]

(b) By any method, find the integral of $g(x, y, z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 2$, $y = 2$, and $z = 2$. [10]

QUESTION 5

(a) By any method, find the circulation of the field $\mathbf{F} = (x^2 + y^2)\hat{\mathbf{i}} + (x + y)\hat{\mathbf{j}}$ around the triangle with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$ traversed in counterclockwise direction. [10]

(b) Verify Green's theorem in the plane for

$$\oint_C [(2xy - x^2)\mathrm{d}x + (x + y^2)\mathrm{d}y],$$

where C is the closed curve (traversed in the counterclockwise direction) of the region bounded by the curves $y = x^2$ and $y^2 = x$. [10]

QUESTION 6

(a) Define the Gamma function, $\Gamma(n)$, where $n > 0$ is a real number. [2]

(b) Show that if n is positive real number, then $\Gamma(n+1) = n!$. [4]

(c) Show that the Gamma function may be defined as

$$\Gamma(n) = \int_0^1 \left(\ln\left(\frac{1}{x}\right)\right)^{n-1} \mathrm{d}x.$$

[5]

(d) Evaluate $\displaystyle\int_0^1 \frac{\mathrm{d}x}{\sqrt{-ln x}}$. [4]

(e) Show that $\displaystyle\int_0^2 x(8-x^3)^{\frac{1}{3}}\mathrm{d}x = \frac{16\pi}{9\sqrt{3}}$. [5]

QUESTION 7

(a) Evaluate

(i) $\displaystyle\int x^4 J_1(x)\mathrm{d}x$ [3]

(ii) $\displaystyle\int x^3 J_3(x)\mathrm{d}x$ [3]

(iii) $\displaystyle\int_0^\infty \frac{y^2}{1+y^4}\mathrm{d}y$. [3]

(iv) $\displaystyle\int_0^\pi \sin^5\theta \mathrm{d}\theta$, [3]

where $J_1(x)$ and $J_3(x)$ are Bessel functions of the first kind of order 1 and order 3, respectively.

(b) Verify Stokes' theorem for $\mathbf{A} = 3y\hat{\mathbf{i}} - xz\hat{\mathbf{j}} + yz^2\hat{\mathbf{k}}$, where S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$ and C is its boundary. [8]

END OF EXAMINATION