University of Swaziland

Final Examination, April/May 2008

BSc III, Bass III, BEd III

Title of Paper : Complex Analysis

Course Number : M313

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) You are given that[1] $z = -1 + i\sqrt{2}$ is a root of the polynomial

$$P(z) = z^4 + z^3 + 2z^2 - z + 3.$$

Find the other 3 roots of $P(z)$, expressing your answers in the form $a + ib$. [8 marks]

(b) Consider the real function $u = e^y(x \cos x + y \sin x)$.

(i) Show that u is harmonic. [4 marks]

(ii) Find the harmonic conjugate of u. [4 marks]

(iii) Hence find the analytic complex function $f(z) = u + iv$ and express in terms of z. [4 marks]

Question 2

(a) State Cauchy's Integral Formula in its most general form. [4 marks]

(b) Prove that

$$\frac{1 - \left(\cos\alpha + i\sin\alpha\right)^6}{1 + \left(\cos\alpha + i\sin\alpha\right)^6} = -i\tan 3\alpha. \qquad [6 \text{ marks}]$$

(c) Use the theory of residues to evaluate

$$\int_0^{2\pi} \frac{d\theta}{\frac{5}{4} + \cos\theta}. \qquad [10 \text{ marks}]$$

[1]Note: Throughout this paper the variable $z = x + iy$ is complex, with real x and y, and $i^2 = -1$.

Question 3

(a) Give a full statement of the Cauchy-Riemann equations in terms of Cartesian coordinates. [4 marks]

(b) The Cauchy-Riemann equations in terms of polar co-ordinates are given by

$$u_r = \tfrac{1}{r}v_\theta, \quad v_r = -\tfrac{1}{r}u_\theta.$$

Use these to show that the formula $f'(z) = u_x + iv_x$ takes the form

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

in polar. [8 marks]

(c) Is the set of values of $\ln\left[\left(1+i\sqrt{3}\right)^3\right]$ the same as that for $3\ln\left(1+i\sqrt{3}\right)$? [8 marks]

Question 4

(a) Evaluate

$$\lim_{z\to 0}\frac{\bar{z}}{z}$$ [4 marks]

(b) Consider the complex function $f(z) = i^z$.

Determine all values of z for which i^z is a purely real number. [8 marks]

(c) Solve for the principal value of

$$\tanh z = i\sqrt{3},$$

and express in the form $a + ib$. [8 marks]

Question 5

(a) Find the Laurent expansion of $f(z) = \dfrac{7}{(2z-1)(z+3)}$ valid in the region

(i) $\frac{1}{2} < |z| < 3$, [5 marks]

(ii) $|z+3| > \frac{7}{2}$. [5 marks]

(b) Consider the complex function

$$\mathcal{H}(z) = \int_\Gamma \frac{\mathrm{Ln} z}{z^2} \mathrm{d}z,$$

where Γ is any circle $|z| = R > 1$ traversed once in the positive sense.

(i) Show that $|\mathcal{H}(z)| < 2\pi\big(\pi + \ln R\big)/R$. [7 marks]

(ii) Hence demonstrate that

$$\lim_{R\to\infty} \mathcal{H}(z) = 0.$$ [3 marks]

Question 6

(a) Use the formula $\sec^{-1} z = -i \ln\left(\dfrac{1+\sqrt{1-z^2}}{z}\right)$ to show that

$$\frac{\mathrm{d}}{\mathrm{d}z}\big(\sec^{-1} z\big) = \frac{1}{z\sqrt{z^2-1}}.$$ [6 marks]

Hence comment about the analyticity of $\sec^{-1} z$.
[2 marks]

(b) Evaluate

$$\int_\Lambda \frac{z^2 - e^{-3iz}}{(2z-\pi)^3} \mathrm{d}z,$$

where Λ is the unit circle $|z| = 1$ traversed once positively.

[6 marks]

(c) Consider the complex function

$$f(z) = \frac{z}{e^z - 1}.$$

(i) Locate and classify all singularities of $f(z)$.

[2 marks]

(ii) Find the value of the residue of $f(z)$ at each of the singularities. [4 marks]

Question 7

(a) Your friend claims that

An essential singularity is a type of a pole.

Is this statement right or wrong? Discuss. [6 marks]

(b) Find the series of

$$f(z) = \frac{1 - \cos^2 z}{z^2}$$

in powers of z. Hence state what type of singularity the point $z = 0$ is. [6 marks]

(c) Evaluate

$$\int_\Omega \frac{\sin \pi z dz}{4z^2 + 1},$$ [8 marks]

where Ω is the unit circle $|z| = 1$ traversed once positively.