UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2007/8

BSc. /B.Ed. /B.A.S.S.

TITLE OF PAPER	:	ABSTRACT ALGEBRA I
COURSE NUMBER	:	M 323
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Prove that a group of prime order has no proper subgroup [6]

   (b) Show that if $(a, m) = 1$ and $(b, m) = 1$ then $(ab, m) = 1 \quad a, b, m \in \mathbb{Z}$ [6]

   (c) Prove that every group of prime order is cyclic [8]

QUESTION 2

2. (a) For each binary operation $*$ defined on a set G, say whether or not $*$ gives a group structure of the set

   i. Define $*$ on $\mathbb{Q}^+$ by $a * b = \frac{ab}{2} \ \forall a, b \in G = Q^+$

   ii. Define on $\mathbb{R}$ by $a * b = ab + a + b \quad \forall a, b \in G = \mathbb{R}$ [10]

   (b) Prove that the binomial coefficient $pC_r = \begin{pmatrix} p \\ r \end{pmatrix}$ with $0 < r < p$ is divisible by the positive prime p. [4]

   (c) Show that $\mathbb{Z}_6$ and S_3 are NOT isomorphic and that $\mathbb{Z}$ and $2\mathbb{Z}$ are isomorphic. [6]

QUESTION 3

3. (a) i. State Cayley's theorem [8]

   ii. Let $(\mathbb{R}^+, 0)$ be the multiplicative group of all positive integers and $(\mathbb{R}, +)$ be the additive group of real numbers. Show that $(\mathbb{R}^+, \cdot)$ is isomorphic to $(\mathbb{R}, +)$ [6]

   (b) Find the number of generators in each of the following

   i. a cyclic group of order 30

   ii. a cyclic group of order 42 [4]

   (c) Determine the right cosets of $H = \{0, 4, 8, 12\}$ in $\mathbb{Z}_{16}$ [6]

QUESTION 4

4. (a) Prove that every subgroup of a cyclic group is cyclic. [10]

   (b) Express $d = (211, 130)$ as an integral linear combination of 211 and 130 [5]

   (c) Solve $3x \equiv 5(mod\ 11)$ [5]

QUESTION 5

5. (a) Prove that a non-abelian group of order $2p, p$ prime contains at least one element of order p. [6]

   (b) Consider the following permutations in S_6

$$\lambda = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \quad \mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$

   Compute

   i. $\lambda\mu$

   ii. μ^2

   iii. μ^{-1}

   iv. μ^{-2}

   v. $\lambda\mu^2$ [10]

   (c) Write the permutations in (b) as a product of disjoint cycles in S_6 [4]

QUESTION 6

6. (a) Define a normal subgroup of a group [4]

   (b) Verify that the subgroup $N = \{(1), (123), (132)\}$ is a normal subgroup of the group S_3 [6]

   (c) For $\mathbb{Z}_{18}$, find all the subgroups and give a lattice diagram [10]

QUESTION 7

7. (a) Let G and H be groups, $\varphi : G \rightarrow H$ be an isomorphism of G and H and let e be the identity of G, prove that $(e)\varphi$ is identity in H and that $[(a)\varphi]^{-1} = (a^{-1})\varphi \quad \forall \ a \in G$ [10]

   (b) Prove that if $(ab)^{-1} = a^{-1}b^{-1} \quad \forall a, b \in G$, where G is a group then G is abelian [5]

   (c) Show that $\mathbb{Z}_p$ has no proper subgroup if p is a prime number. [5]