UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2007/2008

B.Sc. / B.Ed. / B.A.S.S. III

TITLE OF PAPER	:	Real Analysis
COURSE NUMBER	:	M331
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Show that if x and y are real numbers with $x < y$, then there exists an irrational number z such that $x < z < y$. [4]

(b) Let S be a nonempty subset of $\mathbb{R}$. Prove that S has a supremum K if and only if:

(i) $s \leq K$ for all $s \in S$; and

(ii) for every $\varepsilon > 0$, there exists $s_0 \in S$ such that $K - \varepsilon < s_0$. [6]

(c) Let S be a nonempty subset of $\mathbb{R}$ and let a be a real number. Define $a + S = \{a + s : s \in S\}$. Prove that $\sup(a + S) = a + \sup(S)$. [10]

QUESTION 2

(a) (i) Give the definition of a convergent sequence of real numbers. [2]

(ii) Using only the definition, show that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$. [6]

(b) Prove that if a sequence $\{a_n\}$ of real numbers is convergent, then its limit is unique. [6]

(c) Prove that every convergent sequence of real numbers is bounded. Give an example to illustrate why the converse is not always true. [6]

QUESTION 3

(a) (i) What do you understand by a monotone sequence? [2]

(ii) Prove that if a sequence $\{a_n\}$ of real numbers is increasing and bounded above, then it converges to its least upper bound. [5]

(iii) Show that the sequence defined by the recurrence relation

$$y_1 = 1; \qquad y_{n+1} = \frac{1}{4}(2y_n + 3) \, \forall n \in \mathbb{N}$$

is convergent and find its limit. [6]

(b) (i) What is a Cauchy sequence of real numbers? [2]

(ii) Prove that every convergent sequence of real numbers is a Cauchy sequence. [5]

QUESTION 4

(a) (i) Give the Cauchy definition of the limit of a function. [2]

(ii) Using the definition in (i), find

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6.$$

[4]

(b) Let a be an irrational number. Prove that there is a sequence $\{r_n\}$ of rational numbers such that

$$\lim_{n \to \infty} \{r_n\} = a.$$

[4]

(c) (i) Give the Heine definition of the continuity of a function at a point x_0 in its domain. [3]

(ii) Let

$$f(x) = \begin{cases} 2 & \text{if } x \text{ is irrational} \\ -2 & \text{if } x \text{ is rational} \end{cases}$$

Show that $f(x)$ is discontinuous. [7]

QUESTION 5

(a) (i) What do you understand by an open interval? [2]

(ii) Show that the interval $[0,1]$ and the set $\mathbb{Q}$ of rational numbers are not open. [3]

(b) (i) Give an example to illustrate that the intersection of an infinite family of open sets may not be open. [2]

(ii) Prove that any union of open sets is open and that any intersection of closed sets is closed. [5]

(c) Prove that if $\mathcal{H}$ is an open covering of a closed and bounded subset S of $\mathbb{R}$, then S has an open covering $\tilde{\mathcal{H}}$ consisting of finitely many open sets belonging to $\mathcal{H}$. [8]

QUESTION 6

(a) (i) Define the Riemann integral. [4]

(ii) Write down sufficient conditions for a bounded function $f : [0,1] \longrightarrow \mathbb{R}$ to be integrable. [3]

(b Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function. Prove that f is integrable if and only if for any $\varepsilon > 0$, there exists a partition P such that

$$0 \leq U(f,P) - L(f,P) < \varepsilon,$$

where P is a partition of $[a,b]$, $U(f,P)$ is the upper sum of f over P and $L(f,P)$ is the lower sum of f over P. [6]

(c) Let $f : [a,b] \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ -1 & \text{if } x \text{ is rational} \end{cases}$$

Show that f is not Riemann integrable but $|f|$ is. [4]

(d) Define the Riemann-Stietjels integral. [3]

QUESTION 7

(a) Show that

$$\lim_{x \to (0,0)} xy \frac{x^2 - y^2}{x^2 + y^2} = 0.$$

[6]

(b) Show that the function

$$f(x) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$. [8]

(c) Show that the function

$$f(x) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

is not differentiable at $(0,0)$. [6]

END OF EXAMINATION