UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2007/2008

B.Sc. / B.Ed. / B.A.S.S. III

TITLE OF PAPER	:	REAL ANALYSIS
COURSE NUMBER	:	M331
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Prove the triangle inequality in $\mathbb{R}$. [4]

(b) Let A be a set of real numbers which is bounded below. Show that A has only one infimum. [4]

(c) Let A and B be nonempty subsets of $\mathbb{R}$. Prove that if $A \subseteq B$ and B is bounded below, then $\inf(A) \geq \inf(B)$. [6]

(d) State and prove the Archimedean Property of real numbers. [6]

QUESTION 2

(a) (i) What do we mean when we say that a sequence $\{a_n\}$ of real numbers is convergent? [2]

(ii) Using only the definition, show that $\lim_{m\to\infty} \frac{m+2}{2m+3} = \frac{1}{2}$. [6]

(b) State and prove the Squeeze Theorem. [6]

(c) (i) What do you understand by $\lim_{n\to\infty} a_n = +\infty$? [2]

(ii) Show that $\lim_{n\to\infty} \frac{n^2}{2n+2} = +\infty$. [4]

QUESTION 3

(a) (i) What do you understand by an increasing sequence of real numbers? [2]

(ii) Prove that if a sequence $\{a_n\}$ is increasing and bounded above, then

$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$

[5]

(iii) Show that the sequence defined by the recurrence relation

$$x_1 = 1; \qquad x_{n+1} = \sqrt{1+x_n} \qquad \forall n \in \mathbb{N}$$

is convergent and find its limit. [6]

(b) Prove that every Cauchy sequence of real numbers is convergent. [7]

QUESTION 4

(a) What do you understand by uniform continuity of a function f defined on an interval $I \subseteq \mathbb{R}$? [2]

(b) Decide whether or not each of the following functions is uniformly continuous on the given intervals.

(i) $f(x) = \sin\frac{1}{x}$ on $\left(0, \frac{2}{\pi}\right]$. [3]

(ii) $f(x) = \sin x$ on $(-\infty, \infty)$. [3]

(c) Prove that if f is a continuous function on a closed interval $[a, b]$, then f is also uniformly continuous on $[a, b]$. [8]

(d) (i) What is a Lipschitz function? [1]

(ii) Show that if $f : I \longrightarrow \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on I. [3]

QUESTION 5

(a) (i) What do you understand by a limit point of a set $S \subseteq \mathbb{R}$? [2]

(ii) Find all limit points of the set $\mathbb{Q}$ (the set of all rational numbers). [3]

(b) Prove that a set $S \subset \mathbb{R}$ is closed if and only if no point in the complement S^c of S is a limit point of S. [3]

(c) Prove that every infinite bounded set $S \subset \mathbb{R}$ has a limit point. [8]

(d) Prove that every open set in $\mathbb{R}$ is the union of open intervals. [4]

QUESTION 6

(a) Prove that every continuous function in a closed interval $[a, b]$ is integrable. [7]

(b) Let $f : [a, b] \longrightarrow \mathbb{R}$ be an integrable function and let $f = g'$. Show that

$$\int_a^b f \, dx = g(b) - g(a).$$

[8]

(c) Let $f : [0, 1] \longrightarrow \mathbb{R}$ be continuous and let $\varphi : [1, 2] \longrightarrow [0, 1]$ be differentiable. Show that the function $\psi(t) = \int_0^{\varphi(t)} f(x) \, dx$ is differentiable and express $\psi'(t)$ in terms of f and φ'. [5]

QUESTION 7

(a) Find

$$\lim_{x\to(0,0)} \frac{x^3 - y^3}{x^2 + y^2} = 0,$$

if it exists. [6]

(b) Show that the function

$$f(x) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$. [8]

(c) Show that the function

$$f(x) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is not differentiable at $(0,0)$. [6]

END OF EXAMINATION