# UNIVERSITY OF SWAZILAND

## FINAL EXAMINATIONS 2007

## BSc. / BEd. / B.A.S.S. III

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| TITLE OF PAPER | : | DYNAMICS II |
| COURSE NUMBER | : | M 355 |
| TIME ALLOWED | : | THREE (3) HOURS |
| INSTRUCTIONS | : | 1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.<br>2. ANSWER ANY FIVE QUESTIONS |
| SPECIAL REQUIREMENTS | : | NONE |

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

## QUESTION 1

(a) Prove that if the transformation equations are given by $\mathbf{r}_\nu = \mathbf{r}_\nu(q_1, q_2, \ldots, q_n)$ i.e do not involve the time $t$ explicitly, (i) then the kinetic energy can be written as

$$T = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} a_{\alpha\beta} \dot{q}_\alpha \dot{q}_\beta$$

where $a_{\alpha\beta}$ are functions of $q_\alpha$ [5 marks]

and

(ii) $\frac{d}{dt}\left(\frac{\partial r_\nu}{\partial q_\alpha}\right) = \frac{\partial \dot{r}_\nu}{\partial q_\alpha}.$ [5 marks]

(b) For a certain dynamical system the kinetic and potential energy are given by

$$T = \frac{1}{2}\left((1+2k)\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2\right)$$
$$V = \frac{n^2}{2}\left((1+k)\theta^2 + \phi^2\right)$$

where $\theta$ and $\phi$ are generalized coordinates and $n, k$ are positive constants. Write down Lagrange's equations of motion and deduce that

$$(\ddot{\theta} - \ddot{\phi}) + n^2\left(\frac{1+k}{k}\right)(\theta - \phi) = 0.$$

[10marks]

## QUESTION 2

Two particles of masses $m_1$ and $m_2$ are connected by an inextensible string of negligible mass which passes over a fixed frictionless pulley of negligible mass.

(a) Use this system to show that the expressions

(i)

$$H = T + V$$

and (ii)

$$H = \sum p_\alpha \dot{q}_\alpha - L$$

are equivalent. [10marks]

(b) Find the acceleration of $m_1$;

(i) using lagrange's equations. [5marks]

(ii) using Hamilton's equations. [5marks]

## QUESTION 3

(a) Show that if $H$, the Hamiltonian, is independent of time $t$ explicitly, then

(i) $H$ is a constant; [5 marks]

(ii) and that if $\dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} = 2T$, where $L$ is the Lagrangian and $T$ is the kinetic energy then $H$ is equal to the total energy of the system. [4marks]

(b) Consider a dynamical system for the kinetic energy $T$ and potential energy $V$ represented by the expressions

$$T = \frac{1}{2}ma^2(5\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2)$$
$$V = \frac{1}{2}mga(3\theta^2 + \phi^2)$$

where $m, g$ and $a$ are constants and $\theta$ and $\phi$ are the generalized coordinates.

(i) Find an expression for the Hamiltonian of the dynamical system. [6marks]

(ii) Write down Hamilton's equation of motion. [3marks]

(iii) Write down the equations of motion of the system. [2marks]

## QUESTION 4

Prove that the transformation given by $P = \ln \sin p$, $Q = q \tan p$, is a canonical transformation,

(a) by showing that $pdq - PdQ$ is an exact differential, [5marks]]

(b) from first principles. [15marks]

## QUESTION 5

The kinetic energy $T$ and the potential energy $V$ of a system are given by

$$2T = p_1^2 + p_2^2 + p_3^2$$

$$2V = \mu^2(q_1^2 + q_2^2 + q_3^2)$$

Where $q_\alpha$ are the generalized coordinates and $p_\alpha$, $\alpha = 1, 2, 3$ the generalized momenta.

(a) Write down an expression for the Hamiltonian of the system. [2 marks]

(b) Write down Hamilton's equations of motion. [4 marks]

(c) It can be shown that the system has the following six integrals:

$F_1 = q_2 p_3 - q_3 p_2 \qquad G_1 = \mu q_1 \cos \mu t - p_1 \sin \mu t,$

$F_2 = q_3 p_1 - q_1 p_3, \qquad G_2 = \mu q_2 \cos \mu t - p_2 \sin \mu t$

$F_3 = q_1 p_2 - q_2 p_1 \qquad G_3 = \mu q_3 \cos \mu t - p_3 \sin \mu t.$

Find

(i) $\frac{dF_1}{dt}$ [4 marks]

(ii) $[G_1, H]$ [2 marks]

(iii) $[F_3, H]$ [2 marks]

(iv) $[F_1, G_2]$ [2 marks]

(v) $[[F_1, F_3], H]$ [4 marks]

## QUESTION 6

(a) Prove that if the function $F$ in the integral

$$I = \int_a^b F(x, y, y')dx$$

is independent of $x$, then the integral is an extremum if

$$F - y'F_{y'} = c$$

where c is a constant. [5marks]

(b) Using this result or otherwise;

(i) show that the extremum of the integral

$$\int_{y=0}^{y_0} \frac{\sqrt{1+(y')^2}}{\sqrt{y}} dx$$

satisfies the differential equation

$$1 + (y')^2 + 2yy'' = 0$$

[10 marks]

(ii) find the curve which gives an extreme value for the functional

$$I = \int_{x=0}^{1} \left((y')^2 + 1\right) dx$$

when $y(0) = 1$ and $y(1)$ is not specified. [5 marks]

## QUESTION 7

(a) Show that Euler's equation for the functional

$$I = \int_{x=a}^{b} F(x, y, y', y'')dx$$

is given by

$$\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y} = 0.$$

[15marks]

(b) If $[F, G]$ is the poison bracket, prove that

$$\frac{\partial}{\partial t}[F, G] = \left[\frac{\partial F}{\partial t}, G\right] + \left[F, \frac{\partial G}{\partial t}\right].$$

[5marks]

End of Paper