UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2007/8

BSc. / BEd. / B.A.S.S. III

TITLE OF PAPER	:	DYNAMICS II
COURSE NUMBER	:	M 355
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF Seven QUESTIONS. 2. ANSWER ANY Five QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) If the Hamiltonian $H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L$ is expressed as a function of the coordinates q_α and the momenta p_α and the time t, prove that

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \qquad \text{(5 marks)}$$

(b) Consider a dynamical system for which the kinetic energy T and potential energy V are represented by the expressions

$$T = \frac{1}{2}ma^2(5\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 + \dot{q}_2^2), \quad V = \frac{1}{2}mga(3q_1^2 + q_2^2)$$

(i) Find the expression for the Hamiltonian of the system (10 marks)

(ii) Use the Hamiltonian to prove that the equations of motion for the system are

$$\begin{aligned} 4a\ddot{q}_1 + 3gq_1 - gq_2 &= 0 \\ 4a\ddot{q}_2 - 3gq_1 + 5gq_2 &= 0. \end{aligned} \qquad \text{(5 marks)}$$

QUESTION 2

(a) Prove that the transformation given by $P = \ln \sin p$, $Q = q \tan p$, is a canonical transformation by showing that $pdq - PdQ$ is an exact differential, [10marks]

(b) Find the function which optimizes

$$I = \int_{x=0}^{1} (y + y'^2)$$

for the case when

$y(0) = 1, \quad y(1) = 2.$ [10 marks]

QUESTION 3

(a) Given that

$$A_1 = \frac{1}{4}(x^2 + p_x^2 - y^2 - p_y^2), \quad A_2 = \frac{1}{2}(xy + p_x p_y)$$

$$A_3 = \frac{1}{2}(xp_y - yp_x), \quad A_4 = x^2 + y^2 + p_x^2 + p_y^2$$

Evaluate the following Poisson brackets

(i) $[A_1, A_2]$ [3 marks]

(ii) $[A_2, A_3]$ [3 marks]

(iii) $[A_1, A_4]$ [2 marks]

(b) Prove that if the function F in the integral

$$I = \int_a^b F(x, y, y')dx$$

is independent of x, then the integral is an extremum if

$$F - y'F_{y'} = c$$

where c is a constant. [5marks]

(c) Using this result show that the extremum of the integral

$$\int_{y=0}^{y_0} \frac{\sqrt{1 + (y')^2}}{\sqrt{y}} dx$$

satisfies the differential equation

$$1 + (y')^2 + 2yy'' = 0$$

[7 marks]

QUESTION 4

(a) Prove that if the transformation equations are given by $\mathbf{r}_\nu = \mathbf{r}_\nu(q_1, q_2, \ldots, q_n)$ i.e do not involve the time t explicitly, (i) then the kinetic energy can be written as

$$T = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} a_{\alpha\beta} \dot{q}_\alpha \dot{q}_\beta$$

where $a_{\alpha\beta}$ are functions of q_α [5 marks]

and

(ii) $\dfrac{d}{dt}\left(\dfrac{\partial r_\nu}{\partial q_\alpha}\right) = \dfrac{\partial \dot{r}_\nu}{\partial q_\alpha}.$ [5 marks]

(b) For a certain dynamical system the kinetic and potential energy are given by

$$T = \frac{1}{2}\left((1+2k)\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2\right)$$
$$V = \frac{n^2}{2}\left((1+k)\theta^2 + \phi^2\right)$$

where θ and ϕ are generalized coordinates and n, k are positive constants. Write down Lagrange's equations of motion and deduce that

$$(\ddot{\theta} - \ddot{\phi}) + n^2\left(\frac{1+k}{k}\right)(\theta - \phi) = 0.$$

[10marks]

QUESTION 5

(a) A particle of mass m moves in one dimension such that it has the Lagrangian

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2 V(x) - V^2(x),$$

where V is some differentiable function of x. Show that the equation of motion reduces to

$$\left(m\ddot{x} + \frac{dV}{dx}\right)\left(m\dot{x}^2 + 2V(x)\right) = 0.$$

[5 marks]

(b) Show that Euler's equation for the functional

$$I = \int_{x=a}^{b} F(x, y, y', y'')dx$$

is given by

$$\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y} = 0.$$

[15marks]

QUESTION 6

(a) Define the Poisson bracket $[f, g]$ of two dynamical variables which are functions of the generalized coordinates q_α and generalized momenta p_α and time t. [2 marks]

(b) Show that if f is a function of p_α, q_α and t and H is the Hamiltonian, then

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [H, f]$$

[4 marks]

(c) Prove that if the function f does not contain t explicitly, then f is conserved if $[H, f] = 0$. [4 marks]

(d) The Hamiltonian of a two-dimensional harmonic oscillator of unit mass is given by

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2(q_1^2 + q_2^2)$$

where ω is a constant. Given that

$$A = q_1 p_2 - q_2 p_1 \quad \text{and} \quad B = \omega q_1 \sin \omega t + p_1 \cos \omega t$$

(i) Show that both A and B are constants of motion. [7 marks]

(ii) Is $A - B$ a constant of motion ? [3 marks]

QUESTION 7

(a) Show, using first principles that the transformation $P = \frac{1}{2}(p^2+q^2), Q = \arctan(\frac{q}{p})$ is canonical. [13 marks]

(b) The kinetic and potential energy of a certain system are given by

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2\sin^2\alpha)$$

$$V = mgr\cos\alpha$$

where α is a constant. Use the Hamiltonian formulation to show that the angular momentum p_ϕ is conserved and is given by $p_\phi = mh\sin^2\alpha$ where $h = r^2\dot{\phi}$ is a known constant in the theory of forces. [7 marks]