University of Swaziland

Final Examination – December 2007

BSc IV, Bass IV, BEd IV

Title of Paper : Partial Differential Equations
Course Number : M415
Time Allowed : Three (3) hours
Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Consider the expression

$$u - xy = f(x + y^2 - u^2) \qquad (1)$$

where $u = u(x, y)$ and f is an arbitrary function. Find the partial differential equation for which (1) is a general solution. [10 marks]

(b) Use Laplace transforms to solve the system

$$\begin{aligned} u_{xt} - x\cosh t &= 0, & x > 0,\ t > 0, \\ u(x,0) &= x^2, & x \geq 0, \\ u(0,t) &= 0, & t \geq 0. \end{aligned}$$

[10 marks]

Question 2

The electrostatic potential $u(r, \theta)$ inside a capacitor formed by two spheres insulated from each other and maintained at potentials 0 and V_0, respectively, obeys the system

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) = 0, \qquad 0 < r < a,\ 0 < \theta < \pi$$

$$u(a, \theta) = \begin{cases} V_0, & 0 < \theta < \frac{1}{2}\pi \\ 0, & \frac{1}{2}\pi < \theta < \pi \end{cases}$$

Solve for $u(r, \theta)$ inside the capacitor. [20 marks]

Question 3

Consider the PDE

$$25u_{xx} + 20u_{xy} + 4u_{yy} = 24x - y.$$

(a) Classify the PDE as hyperbolic, parabolic or elliptic. [4 marks]

(b) Reduce the equation into its canonical form and hence find its general solution. [16 marks]

Question 4

Consider the non-homogeneous boundary-value problem

$$\begin{aligned} u_t - u_{xx} &= e^{-t}\sin(\tfrac{1}{2}x), & 0 < x < \pi,\ t > 0, \\ u(x,0) &= 2\sin(\tfrac{3}{2}x), & 0 \le x \le \pi, \\ u(0,t) &= u_x(\pi,t) = 0, & t \ge 0. \end{aligned}$$

(a) Show that the solution of the associated homogeneous problem is of the form $\sum_{n=1}^{\infty} u_n(t)\sin\alpha_n x$, where $\alpha_n = \frac{2n-1}{2}$. [7 marks]

(b) Hence, or otherwise, solve the non-homogeneous problem.

[13 marks]

Question 5

Find the solution of the steady-state problem [20 marks]

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & 0 < x < \pi,\ 0 < y < \pi, \\ u(0,y) &= 8\sin^3 y, & 0 \le y \le \pi, \\ u(\pi,y) &= 0, & 0 \le y \le \pi, \\ u(x,0) &= u(x,\pi) = 0, & 0 \le x \le \pi. \end{aligned}$$

Question 6

(a) Find the particular solution of the PDE

$$x(y-u)u_x + y(u-x)u_y = u(x-y)$$

which contains the straight line $x = y = u$. [10 marks]

(b) Find the particular solution of the PDE

$$xu_{xy} + u_y = 2x$$

satisfying the condition: $u = x,\ u_y = 1$ when $x - y = 0$.
[10 marks]

Question 7

Solve the Dirichlet problem inside the circle

$$\begin{aligned} u_{rr} + \tfrac{1}{r}u_r + \tfrac{1}{r^2}u_\theta &= 0, \quad 0r < 1, \quad -\pi < \theta < \pi, \\ u(1,\theta) &= 1 - \cos 2\theta, \quad -\pi \le \theta \le \pi. \end{aligned}$$

[20 marks]

Table of Laplace Transforms

$f(t)$	$F(s)$
t^n	$\frac{n!}{s^{n+1}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\frac{1}{a-b}(e^{at}-e^{bt})$	$\frac{1}{(s-a)(s-b)}$
$\frac{1}{a-b}(ae^{at}-be^{bt})$	$\frac{s}{(s-a)(s-b)}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at) - at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sin(at)\sinh(at)$	$\frac{2a^2}{s^4+4a^4}$
$\sinh(at)\sin(at)$	$\frac{2a^3}{s^4-a^4}$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \ldots - f^{n-1}(0)$