University of Swaziland

Supplementary Examination 2008

BSc IV, Bass IV, BEd IV

Title of Paper : Partial Differential Equations
Course Number : M415
Time Allowed : Three (3) hours
Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Consider the expression

$$u = xy + f(x - y + u) \qquad (1)$$

where $u = u(x, y)$ and f is an arbitrary function.
Find the partial differential equation for which (1) is a general solution. [10 marks]

(b) Use Laplace transforms to solve the system

$$\begin{aligned} u_{xt} - 2x\sin t &= 0, \qquad x > 0,\ t > 0, \\ u(x,0) &= x, \qquad x \geq 0, \\ u(0,t) &= \cos t, \quad t \geq 0. \end{aligned}$$

[10 marks]

Question 2

Consider the Cauchy problem for the wave equation

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad -\infty < x < \infty,\ t \geq 0, \\ u(x,0) &= f(x), \quad -\infty < x < \infty, \\ u_t(x,0) &= g(x), \quad -\infty < x < \infty. \end{aligned}$$

Derive the d'Alembert's solution

$$u(x,t) = \frac{1}{2}\Big\{f(x+ct) + g(x-ct)\Big\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\alpha)\mathrm{d}\alpha.$$

[20 marks]

Question 3

Consider the PDE

$$2u_{xx} - 10u_{xy} + 8u_{yy} + u_x - u_y = 0.$$

(a) Classify it as hyperbolic, parabolic or elliptic.

[4 marks]

(b) Reduce the equation into its canonical form and hence find its general solution. [16 marks]

Question 4

Solve the boundary-value problem

$$\begin{aligned} u_t - u_{xx} &= 0, \quad 0 < x < \pi,\ t > 0, \\ u(x,0) &= \cos x, \quad 0 \le x \le \pi, \\ u_x(0,t) &= u(\pi,t) = 0, \quad t \ge 0. \end{aligned}$$

[20 marks]

Question 5

Find the solution of the steady-state problem [20 marks]

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad 0 < x < \pi,\ 0 < y < \pi, \\ u(x,0) &= 0, \quad 0 \le x \le \pi, \\ u(x,\pi) &= 6\sin 2x - 4\sin 4x, \quad 0 \le x \le \pi, \\ u(0,y) &= u(\pi,y) = 0,\ 0 \le y \le \pi. \end{aligned}$$

Question 6

(a) Find the particular solution of the PDE

$$yu_x + xu_y = u(x-y)$$

which contains the straight curve $u = 1$ on $y = x^2$. [10 marks]

(b) Find the particular solution of the PDE

$$u_{xy} = 4xy$$

satisfying the condition: $u = x,\ u_y = 1$ when $x - y = 0$.

[10 marks]

Question 7

The initial temperature distribution of a thin circular disk is given by T_0. If the disk is allowed to cool down with its circular edge kept at temperature zero, the subsequent temperature distribution satisfies the system

$$\begin{aligned} k\left(u_{rr} + \tfrac{1}{r}u_r\right) &= u_t, & 0 < r < 1,\ t > 0, \\ u(r, 0 &= T_0, & 0 \le r \le 1, \\ u(1, t) &= 0, & t \ge 0. \end{aligned}$$

Solve for $u(r, t)$.

[20 marks]

Table of Laplace Transforms

$f(t)$	$F(s)$
t^n	$\frac{n!}{s^{n+1}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\frac{1}{a-b}(e^{at} - e^{bt})$	$\frac{1}{(s-a)(s-b)}$
$\frac{1}{a-b}(ae^{at} - be^{bt})$	$\frac{s}{(s-a)(s-b)}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at) - at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sin(at)\sinh(at)$	$\frac{2a^2}{s^4+4a^4}$
$\sinh(at)\sin(at)$	$\frac{2a^3}{s^4-a^4}$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \ldots - f^{n-1}(0)$