UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2007/8

BSc. /B.Ed. /B.A.S.S.

TITLE OF PAPER	:	ABSTRACT ALGEBRA
COURSE NUMBER	:	M 423
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Find all the monic irreducible polynomials of degree 2 over $\mathbb{Z}_3$ [9]

   (b) Prove that every field is an integral domain [7]

   (c) State the Eisentein Criterion for irreducibility. [4]

QUESTION 2

2. (a) Which of the following are rings with the usual addition and multiplication

      i. $\left\{a + b\sqrt{2}, \quad a, b \in \mathbb{Z}\right\}$ [5]

      ii. $M_2(\mathbb{R})$ with zero determinant [5]

   (b) Factor the polynomial $4x^2 - 4x + 8$ as a product of irreducibles viewing it as an element of the following integral domain

      i. $\mathbb{Z}[x]$

      ii. $\mathbb{Z}_{11}[x]$ [10]

QUESTION 3

3. (a) Find the greatest common divisor of the polynomials

$$f(x) = x^4 + 4x^3 + 7x^2 + 6x + 2$$

$$g(x) = x^3 + 4x^2 + 7x + 4$$

   Over Q and express it as a linear combination of $f(x)$ and $g(x)$ [8]

   (b) Prove that if R is a ring with unity and N is an ideal of R containing a unit, then $N = R$ [6]

   (c) Describe all units in each of the following rings

      i. $\mathbb{Z}_7$

      ii. $\mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}$ [6]

QUESTION 4

4. (a) State whether or not each of the given function v is an Euclidean valuation for the given integral domain.

   i. The function v for $\mathbb{Z}$ given by $v(n) = n^2$ for non-zero $n \in \mathbb{Z}$

   ii. The function of v of $\mathbb{Q}$ given by $v(a) = 50$ for non-zero values $a \in \mathbb{Q}$ [8]

   (b) Given that every element β of $E = F(\alpha)$ can be uniquely expressed in the form $\beta = b_0 + b_1\alpha + b_2\alpha^2 + \cdots + b\alpha^{n-1}$ where each $b_i \in F, \alpha$ algebraic over the field F^{n-1} and degree $(\alpha, F) \geq 1$. Show that if F is finite with q elements, then $F(\alpha)$ has q^n elements [6]

   (c) State Kronecker's theorem. [Do not prove] [6]

QUESTION 5

5. (a) Prove that, if D is an integral domain, then $D[x]$ is also an integral domain [6]

   (b) Use Fermats theorem to compute the remainder when 8^{103} is divided by 13. [6]

   (c) For each of the given algebraic number $\alpha \in C$, find irred (α, Q) and $\deg(\alpha, Q)$

   i. $\sqrt{\frac{1}{3} + \sqrt{7}}$

   ii. $\sqrt{2} + i$ [8]

QUESTION 6

6. (a) Determine which of the following polynomial in $\mathbb{Z}[x]$ satisfy Eisenstein criterion for irreducibility over Q

   i. $4x^{10} - 9x^3 + 24x - 18$

   ii. $2x^{10} + 25x^3 + 10x^2 - 30$ [10]

   (b) Prove that every finite integral domain is a field. [10]

QUESTION 7

7. (a) Show that for a field F, the set of all matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix} \quad a_{ij} \in F$$

   is a right ideal but not a left ideal of $M_2(F)$. [6]

   (b) Let $\varphi_\alpha : \mathbb{Z}_7[x] \to \mathbb{Z}_7$. Evaluate each of the following for the indicated evaluation homomorphism

   i. $\varphi_5\left[(x^3+2)(4x^2+3)(x^7+3x^2+1)\right]$ [5]

   ii. $\varphi\left[3x^{106} + 5x^{99} + 2x^{53}\right]$ [5]

   (c) Show that the rings $3\mathbb{Z}$ and $5\mathbb{Z}$ are not isomorphic [5]