UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2007/2008

B.Sc. / B.Ed. / B.A.S.S. IV

TITLE OF PAPER	:	Metric Spaces
COURSE NUMBER	:	M431
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Let X be a nonempty set.

(i) What is meant by saying that (X, d) is a *metric space*?

(ii) Let d be the function on $\mathbb{R}^2$ defined by

$$d(x, y) = c_1|x_1 - y_1| + c_2|x_2 - y_2|,$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$, and $c_i > 0$. Prove that $(\mathbb{R}^2, d)$ is a metric space. [10]

(b) Let $A = \{(x_1, x_2) : 0 \leq x_1, 0 \leq x_2, x_1 + x_2 \leq 2\}$ and let $x = (2, 2)$. Find $d(x, A)$ for the Euclidean, the Max, and for the New York metrics. (Recall that the New York metric is defined by

$$d(x, y) = \begin{cases} |y_2 - x_2| & \text{if} \quad x_1 = y_1 \\ |x_2| + |y_1 - x_1| + |y_2| & \text{if} \quad x_1 \neq y_1. \end{cases}\Bigg)$$

Calculate $\operatorname{diam}(A)$ in each case. [10]

QUESTION 2

(a) Let (X, d) be a metric space and let $S \subseteq X$. What is meant by saying that S is closed? Prove that any intersection of closed sets in X is closed and any finite union of closed sets in X is closed. [8]

(b) What is meant by an *open ball* $B(a, r)$ in a metric space (X, d)? Show that an open ball is open. By drawing a diagram, or otherwise, describe the open ball $B(a, 3)$ in $\mathbb{R}^2$, where $a = (4, 5)$

(i) with the usual metric

(ii) with the max metric. [6]

(c) Prove that in any metric space X, each closed ball is a closed set. Show that any finite set in X is closed [6]

QUESTION 3

(a) Prove that:

(i) Every convergent sequence is a Cauchy sequence; [3]

(ii) If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in a metric space (X, d), then the sequence $\{d(x_n, y_n)\}$ is convergent in $\mathbb{R}$. [6]

(b) What do you understand by the following:

(i) A nowhere dense metric space; [2]

(ii) An everywhere dense metric space. [1]

(c) State and prove Baire's theorem. [8]

QUESTION 4

(a) Given a function $f : (X, d_1) \longrightarrow (X, d_2)$,

(i) When is f said to be continuous in the $\varepsilon - \delta$ sense?

(ii) Give an equivalent definition in terms of open sets.

(iii) Assuming f is continuous at x_0, prove that

$$x_n \to x_0 \Rightarrow f(x_n) \to f(x_0).$$

[14]

(b) Prove that the function $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $\pi(x, y) = x$ is continuous when $\mathbb{R}^2$ and $\mathbb{R}$ are equipped with their usual metrics. Is π uniformly continuous? Justify your answer. [6]

QUESTION 5

(a) Define compactness of a metric space in terms of

(i) open coverings,

(ii) sequences. [5]

(b) Prove that any closed subspace of a compact metric space is compact. [7]

(c) prove that the continuous image of a compact subset of a metric space is compact. [8]

QUESTION 6

(a) When are two subsets A and B of a metric space said to be separated? [2]

(b) Verify that two nonempty disjoint closed sets in a metric space are separated. [2]

(c) Give two alternate definitions of connectedness of a subset M of a metric space X. [4]

(d) (i) Prove that if X is a connected metric space and $f : X \longrightarrow \mathbb{R}$ is a continuous function, then $f(X)$ is connected.

(ii) Deduce that if $f : [0,1] \longrightarrow [0,1]$ is continuous, then there exists an $x \in [0,1]$ such that $f(x) = x$. [12]

QUESTION 7

(a) (i) What is a Lebesgue number for a given open cover of a metric space? [2]

(ii) State and prove Lebesgue's Covering Lemma. [8]

(b) (i) Explain what is meant by a contraction of a metric space, and state without proof the Contraction Mapping Theorem.

(ii) Show that the mapping $f : [-1,1] \longrightarrow [-1,1]$ defined by $f(x) = \frac{1}{14}(x^4 - 3x^3 + 9)$ is a contraction, and deduce that there is unique solution to the equation $x^4 - 3x^3 - 14x + 9 = 0$ in the interval $[-1,1]$. [10]

END OF EXAMINATION