UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2007/2008

B.Sc. / B.Ed. / B.A.S.S. IV

TITLE OF PAPER	:	Metric Spaces
COURSE NUMBER	:	M431
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Let X be a nonempty set.

(i) What is meant by saying that (X, d) is a *metric space*?

(ii) Let d be the function on $\mathbb{R}^2$ defined by

$$d(x, y) = \max\{3|x_1 - y_1|, 2|x_2 - y_2|\},$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$. Prove that $(\mathbb{R}^2, d)$ is a metric space. [10]

(b) Let $A = \{(x_1, x_2) : 0 \leq x_1, 0 \leq x_2, x_1 + x_2 \leq 2\}$ and let $x = (2, 2)$. Find $d(x, A)$ for the Euclidean, the Max, and for the New York metrics. (Recall that the New York metric is defined by

$$d(x, y) = \begin{cases} |y_2 - x_2| & \text{if } \ x_1 = y_1 \\ |x_2| + |y_1 - x_1| + |y_2| & \text{if } \ x_1 \neq y_1. \end{cases}\Bigg)$$

Calculate $\operatorname{diam}(A)$ in each case. [10]

QUESTION 2

(a) Let (X, d) be a metric space and let $S \subseteq X$. What is meant by saying that S is open? Prove that any union of open sets in X is open and any finite intersection of open sets in X is open. [8]

(b) What is meant by an *open ball* $B(a, r)$ in a metric space (X, d)? Show that an open ball is open. By drawing a diagram, or otherwise, describe the open ball $B(a, 3)$ in $\mathbb{R}^2$, where $a = (2, 5)$

(i) with the usual metric,

(ii) with the max metric. [6]

(c) Show that $\emptyset$ and X are closed, where (X, d) is a metric space. [6]

QUESTION 3

(a) Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X. What is meant by saying that $\{x_n\}$ is *convergent*? [2]

(b) Decide whether or not the following sequences are convergent in the usual (Euclidean) metric on $\mathbb{R}^2$:

(i) $x_n = \left(\dfrac{n^3}{3n^3+1}, \dfrac{3}{3n+3}\sin(\dfrac{n\pi}{2})\right)$,

(ii) $x_n = (10^{-n}, (-1)^n \exp(\dfrac{1}{n}))$. [8]

(c) (i) Suppose that $\{x_n\}$ converges to x in $C[a, b]$ in the uniform metric. Explain what is meant by *pointwise convergence* of a sequence $\{x_n\}$ in $C[a, b]$. Show that $\{x_n\}$ converges to x pointwise.

(ii) Let x_n in $C[0, 1]$ be defined by

$$x_n(t) = \begin{cases} \frac{nt}{n-1} & \text{if } 0 \le t \le 1 - \frac{1}{n}, \\ n(1-t) & \text{if } 1 - \frac{1}{n} \le t \le 1. \end{cases}$$

Sketch the graph of $x_n(t)$ and show that $\{x_n\}$ converges pointwise to the function

$$x(t) = \begin{cases} t & \text{if } 0 \le t < 1, \\ 0 & \text{if } t = 1. \end{cases}$$

Deduce that $\{x_n\}$ is not convergent in $C[0, 1]$ in the uniform metric. [10]

QUESTION 4

(a) Prove that in a metric space (X, d), a subset $F \subseteq X$ is closed if the limit of any convergent sequence $\{x_n\}$ of points of F is in F. [8]

(b) Prove that $\mathbb{R}^2$ equipped with the metric

$$d(x, y) = \alpha|x_1 - y_1| + |x_2 - y_2|, \qquad x = (x_1, x_2), \quad y = (y_1, y_2)$$

is complete, where $\alpha > 0$ is fixed. [12]

QUESTION 5

(a) Let X be a metric space. When is a subset $M \subseteq X$ said to be:

(i) bounded;

(ii) totally bounded. [3]

(b) Prove that in $\mathbb{R}$ with the usual metric, the notions of boundedness and total boundedness are equivalent. [5]

(c) Show that a compact set is closed and bounded. [6]

(d) Which of the following sets is compact? Give reasons.

(i) $\{(x, y) : 0 \leq x \leq y \leq 1\}$ in $\mathbb{R}^2$,

(ii) $\{1, \frac{1}{3}, \frac{1}{3^2}, \ldots, \frac{1}{3^n}, \ldots\}$ in $\mathbb{R}$, where $n \in \mathbb{N}$. [6]

QUESTION 6

(a) When are two subsets A and B of a metric space said to be separated? [2]

(b) Verify that two nonempty disjoint closed sets in a metric space are separated. [2]

(c) Give two alternate definitions of connectedness of a subset M of a metric space X. [4]

(d) (i) Prove that if X is a connected metric space and $f : X \longrightarrow \mathbb{R}$ is a continuous function, then $f(X)$ is connected.

(ii) Deduce that if $f : [0,1] \longrightarrow [0,1]$ is continuous, then there exists an $x \in [0,1]$ such that $f(x) = x$. [12]

QUESTION 7

(a) (i) What is a Lebesgue number for a given open cover of a metric space?[2]

(ii) State and prove Lebesgue's Covering Lemma. [8]

(b) (i) Explain what is meant by a contraction of a metric space, and state without proof the Contraction Mapping Theorem.

(ii) Show that the mapping $f : [-1,1] \longrightarrow [-1,1]$ defined by $f(x) = \frac{1}{12}(x^5 - 2x^3 + 8)$ is a contraction, and deduce that there is unique solution to the equation $x^5 - 2x^3 - 12x + 8 = 0$ in the interval $[-1,1]$. [10]

END OF EXAMINATION