UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2008/9

BSc. /BEd. /B.A.S.S II

TITLE OF PAPER	:	LINEAR ALGEBRA
COURSE NUMBER	:	M 220
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
		2. ANSWER ANY FIVE QUESTIONS.
		3. Non-programmable calculators may be used.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Verify which of the following are linear transformations.

   (i) $T : \mathbb{R}^2 \to \mathbb{R}^3;\ T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ 3y \\ 2x-y \end{pmatrix}.$ [5 marks]

   (ii) $T : \mathbb{R}^2 \to \mathbb{R}^2;\ T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ y \end{pmatrix}.$ [5 marks]

   (b) Find standard matrices for the following linear transformations.

   (i) $T : \mathbb{R}^3 \to \mathbb{R}^2;\ T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+z \\ y-z \end{pmatrix}.$ [5 marks]

   (ii) $T : \mathbb{R}^3 \to \mathbb{R}^4;\ T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ x+z \end{pmatrix}.$ [5 marks]

QUESTION 2

2. (a) Define a vector space. [6 marks]

   (b) Find the inverses of the following matrices.

   (i) $\begin{pmatrix} 2 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$

   (ii) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}.$ [6 marks]

   (c) (i) Use the inverse of 2(b)i to solve the following system of equations

$$\begin{aligned} 2x + 2y + z &= 1 \\ 3x + y + z &= 2 \\ x + y + z &= 2 \end{aligned}$$

   (ii) Use Cramer's rule to solve the following system of equations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ x_1 + x_2 - x_3 &= 1 \\ x_1 - x_2 - x_3 &= 1 \end{aligned}$$

[8 marks]

QUESTION 3

3. (a) For which k does the following system have a trivial solution?

$$\begin{aligned} kx + y - 3z &= 0 \\ (k-1)x + ky + z &= 0 \\ 3x + (k-1)y + kz &= 0 \end{aligned}$$

[8 marks]

(b) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}$$

[12 marks]

QUESTION 4

4. (a) Find the invserse A^{-1} of the matrix A using the augmented matrix $[A : I]$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{pmatrix}$$

[5 marks]

(b) Using 4a above find a sequence of elementary matrices $E_1, E_2, \ldots, E_r$ such that $E_1E_2 \ldots E_rA = I$, i.e $A = E_r^{-1}, E_{r-1}^{-1}, \ldots, E_2^{-1}, E_1^{-1}$. [5 marks]

(c) Write A and A^{-1} as a product of elementary matrices. [5 marks]

(d) Determine whether the following system has a non-trivial solution.

$$\begin{aligned} x + y + z + w &= 0 \\ 2x + y - z + 2w &= 0 \\ 3x + 2y + 2z + 2w &= 0 \end{aligned}$$

[5 marks]

QUESTION 5

5. (a) Prove that if A and B are both non-singular $n \times n$ matrices then AB is non-singular and $(AB)^{-1} = B^{-1}A^{-1}$. [8 marks]

   (b) Use Cramer's rule to solve the system

$$\begin{aligned} x - 3y + z &= -1 \\ -2x + 2y - z &= 1 \\ 4x - 4y + z &= -2 \end{aligned}$$

   (c) Use Gaussian elimination to solve the system

$$\begin{aligned} x_1 + x_2 + 2x_3 + 3x_4 &= 13 \\ x_1 - 2x_2 + x_3 + x_4 &= 8 \\ 3x_1 + x_2 + x_3 - x_4 &= 1 \end{aligned}$$

QUESTION 6

6. (a) Prove that if a homogeneous system has more unknowns than the number of equations, then it always has a non-trivial solution. [10 marks]

   (b) Let $B = \{v_1, v_2, v_3\}$ and $B' = \{u_1, u_2, u_3\}$ be bases in $\mathbb{R}^3$, where

$$v_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

   Find the transition matrix from B to B'. [10 marks]

QUESTION 7

7. (a) Verify the Cayley-Hamilton theorem for the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 5 \\ 3 & 2 & 1 \end{pmatrix}$$

[5 marks]

(b) Let $S = \{u_1, u_2, \ldots, u_n\}$ be a set of non-zero vector in the a vector space V. Prove that S is linearly independent if and only if one of the vectors u_j is a linear combination of the preceeding vectors in S. [10 marks]

(c) Show that each eigenvector of the square matrix A is associated with only one eigenvector. [5 marks]