UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2008/9

BSc. / BEd. / B.A.S.S. III

TITLE OF PAPER	:	NUMERICAL ANALYSIS I
COURSE NUMBER	:	M 311
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Convert the decimal 9.7 into its binary equivalent. [5 marks]

   (b) Convert the binary $(0.\overline{10})_2$ into its decimal equivalent. [5 marks]

   (c) Determine the machine representation in single precision on a 32-bit word length computer (Marc-32) for the decimal number -285.75 [5 marks]

   (d) Find the roots of the following quadratic equation (as accurately as possible) using eight digits and rounding [5 marks]

$$x^2 - 100000x + 1 = 0$$

QUESTION 2

2. (a) Given the function $f(x) = \dfrac{e^x - 1 - x}{x^2}$

      i. find a suitable function $g(x)$ that has been reformulated to be algebraically equivalent to $f(x)$ with the aim of avoiding loss of significance error. [5 marks]

      ii. Compare the results of calculating $f(0.01)$ and $g(0.01)$ using six digits and rounding. [5 marks]

   (b) Given the initial interval [2,5] for the Bisection Method, how many iterations are needed to guarantee that our solution is accurate to 10^{-10}. [4 marks]

   (c) Consider the iteration

$$x_{n+1} = 2x_n - \alpha x_n^2, \qquad n = 0, 1, \ldots$$

   where $\alpha > 0$ is given. Show that the iteration converges quadratically to $1/\alpha$ for any initial guess x_0. [6 marks]

QUESTION 3

3. (a) Use the Lagrange Interpolation polynomial to interpolate $f(x)$ from the following table:

x	0	1	2	4
$f(x)$	7	13	21	43

and find an approximation to $f(3)$ [7 marks]

(b) The population of a city in a census taken once in ten years is given by

Year	1921	1931	1941	1951
Population in Thousands	35	42	58	84

i. Construct a forward divided difference table for the above tabulated data. [3 marks]

ii. Use Newton interpolation formula of degree 2 to obtain (approximately) the population of the city in 1925. [4 marks]

(c) i. Show that the function

$$f(x) = e^x - x^2$$

has exactly one zero in the interval [-1, 0]. [2 marks]

ii. Use 4 iterations of the Newton-Raphson method with an initial guess of $x_0 = 0$ to obtain an approximation to this root. [4 marks]

QUESTION 4

4. (a) For the function $f(x) = \ln(x+1)$, let $x_0 = 0, x_1 = 0.6$, and $x_2 = 0.9$. Construct the Lagrange interpolating polynomial of degree at most 2 to approximate $f(0.45)$, and find the actual error. [6 marks]

(b) Suppose we know the following values of a function f:

$$f(0) = 1, f(1) = 2, f(2) = 8$$

i. Evaluate the divided-differences $f[0], f[0,1], f[0,1,2]$. [3 marks]

ii. Evaluate the forward-differences $\Delta f(x_0), \Delta^2 f(x_0)$. [4 marks]

iii. Write down the appropriate Newton's interpolating polynomial. [3 marks]

(c) For a function $f(x)$ the forward divided-differences are given by

$x_0 = 1.0$	$f[x_0] = \square$		
		$f[x_0, x_1] = 3$	
$x_1 = 2.0$	$f[x_1] = 11$		$f[x_0, x_1, x_2] = 1$
		$f[x_1, x_2] = \square$	
$x_2 = 4.0$	$f[x_2] = \square$		

Determine the missing entries in the table. [4 marks]

QUESTION 5

5. (a) Consider the integral $I = \int_0^1 \sqrt{2-x}\, dx$.

i. Find the exact value of the Integral. [4 marks]

ii. Use the trapezoidal rule with five subintervals to approximate the integral and compare your result against the exact value of the integral. [6 marks]

(b) How large should n be so that the trapezoidal approximation to the integral $\int_0^2 \frac{1}{x+1}\, dx$ is accurate to within 0.00001? [4 marks]

(c) The quadrature formula $\int_{-1}^{1} f(x)\, dx \approx c_0 f(-1) + c_1 f(0) + c_2 f(1)$ is exact for all polynomials of degree less than or equal to 2. Determine c_0, c_1 and c_2. [6 marks]

QUESTION 6

6. (a) Given that a is the fixed point of the following iteration scheme (for all $a \neq -5$.)

$$\alpha_{n+1} = \frac{\alpha_n^2 - a\alpha_n + a^2 + 5a}{\alpha_n + 5}$$

Find the order of convergence and the asymptotic error constant. [4 marks]

(b) The positive root of $f(x) = \alpha - \beta x^2 - x$ with $\alpha,\ \beta > 0$ is sought and the simple iteration

$$x_{n+1} = \alpha - \beta x_n^2$$

is used. Show that convergence will occur for sufficiently close starting value, provided

$$\alpha\beta < \frac{3}{4}$$

[6 marks]

(c) Consider the linear system $A\mathbf{x} = \mathbf{b}$ where,

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -1 & 1 \\ 2 & -2 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Use any method to find the LU decomposition of matrix A and then solve the system. [10 marks]

QUESTION 7

7. (a) Find an approximate value of $\int_1^2 \frac{1}{x}\, dx$ using composite Simpson's rule with $h = 0.25$ and evaluate the bound on the error.

[10 marks]

(b) Suppose that

$$\mathbf{x} = \begin{pmatrix} -0.17 \\ 0.22 \end{pmatrix}$$

is an approximate solution of the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 5 & 7 \\ 7 & 10 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0.7 \\ 1 \end{pmatrix}$$

i. Discuss the ill-conditioning of the system. [4 marks]

ii. Compute the residual vector $\mathbf{r}$ and then find the upper bound for a relative error in solving the given linear system. [6 marks]