UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2008/09

BSc.III

TITLE OF PAPER	:	VECTOR ANALYSIS
COURSE NUMBER	:	M312
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE (5) QUESTIONS 3. ONLY NON-PROGRAMMABLE CALCULATORS MAY BE USED.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Consider a polar coordinate system (r, θ). Derive formulas for the

(i) Velocity,

(ii) Acceleration.

(b) Given $r = e^{3t}, \quad \theta = 4t$ in the polar coordinate system. Find the

(i) trajectory,

(ii) radius of curvature of the trajectory.

(c) (i) Define a sectorial velocity and

(ii) find it in polar coordinate system.

(d) Prove that a plane polar coordinate system is orthogonal.

[2,5,1,4,1,3,4]

QUESTION 2

(a) Consider curvilinear coordinates q_1, q_2, q_3 in general case.

(i) Define Lame parameters H_i.

(ii) Show that

$$H_i^2 = \left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2.$$

(b) Let $\overline{e}_i$ and $\overline{e}_j$ be unit vectors corresponding to the coordinates q_i and q_j respectively.

(i) Find the angle between $\overline{e}_i$ and the coordinate axes OX, OY, OZ.

(ii) Find angle between $\overline{e}_i$ and $\overline{e}_j$, and hence

(iii) Derive a condition of orthogonality of $\overline{e}_i$ and $\overline{e}_j$.

(c) Consider a system of spherical coordinates.

(i) Find Lame parameters.

(ii) Show that the system is orthogonal. [2,2,4,3,2,3,4]

QUESTION 3

(a) Consider a system of curvilinear orthogonal coordinates. Derive formulas for the

(i) Velocity,

(ii) Acceleration,

(b) Prove that grad f is perpendicular to the surface $f = const.$

(c) Consider a function $f(x, y, z) = x^2yz + 4xz^2$ and a point $\underline{P}(1, -2, -1)$.

(i) Find the directional derivative of f at $\underline{P}$ in the direction of the vector $\overline{A}(2, -1, -1)$.

(ii) Compute the greatest rate of change of f at $\underline{P}$.

(iii) Find the direction of the maximum rate of increase of f at $\underline{P}$.

(d) Find the unit normal to the surface $(x-1)^2 + y^2 + (z+2)^2 = 9$ at $\underline{P}(3, 1, -4)$.

[2,5,3,3,2,2,3]

QUESTION 4

(a) Prove the following formulae;

(i) $\nabla \cdot (f\overline{A}) = f\nabla \cdot \overline{A} + \nabla f \cdot \overline{A}$,

(ii) $\nabla \times (\nabla f) = 0$,

(iii)$\nabla \cdot (\nabla \times \overline{A}) = 0$,

(iv) $(\overline{A} \cdot \nabla)\overline{r} = \overline{A}$.

(b) Give the definition of a

(i) regular arc,

(ii) regular curve,

(iii) Line integral of $\overline{u}$ over curve c.

(c) The acceleration of a particle at any time $t \geq 0$ is given by

$$\overline{a} = 12\cos 2t\overline{i} - 8\sin 2t\overline{j} + 16t\overline{k}.$$

The position vector and velocity are zero at $t = 0$. Find at any time the

(i) Velocity, and the

(ii) position vector.

[3,3,4,3,1,1,2,2,1]

QUESTION 5

(a) Evaluate the line integral for

$\overline{U} = x^2\overline{i} + y\overline{j}$ along the parabola $y = x^2$ from the origin to the point $(1, 1)$:

(i) First by considering the parameter equation of the parabola $x = t$ and $y = t^2$.

(ii) Next by considering the parabola in rectangular form.

(b) Is $\overline{U}$ in (a) conservative? Explain.

(c) (i) State, and

(ii) prove the necessary and sufficient condition theorem that a line integral in a simply connected region is independent of the path from point A to point B.

[4,4,3,2,7]

QUESTION 6

(a) Show that a

(i) necessary, and

(ii) sufficient condition that $\overline{U} = \nabla\phi$ is $rot\overline{U} = 0$.

(b) Determine the constant a so that the vector $\overline{A} = (x+3y)\overline{i} + (y-2z)\overline{j} + (x+az)\overline{k}$ is solenoidal.

(c) Consider a vector field

$$\overline{V} = (x + 2y + az)\overline{i} + (bx - 3y - z)\overline{j} + (4x + cy + 2z)\overline{k}.$$

(i) Find constants a, b, c, so that the field is irrotational.

(ii) Find a scalar potential of $\overline{V}$.

[3,4,3,3,7]

QUESTION 7

(a) Evaluate $\int_c (y - \sin x)dx + \cos x dy$,

where c is the triangle with the vertices at $O, A\left(\frac{\pi}{2}, 0\right), B\left(\frac{\pi}{2}, 1\right)$

(i) Directly,

(ii) by using Green's theorem in the plane.

(b) State

(i) Divergence theorem,

(ii) Stoke's theorem.

(c) Prove that a necessary and sufficient condition that

$$\int_c \overline{A} \cdot d\overline{r} = 0$$

for every closed curve c is that

$$\nabla \times \overline{A} = 0$$

identically. [4,4,3,3,6]