# UNIVERSITY OF SWAZILAND

## SUPPLEMENTARY EXAMINATION 2008/09

## BSc.III

---

| | | |
|---|---|---|
| TITLE OF PAPER | : | VECTOR ANALYSIS |
| COURSE NUMBER | : | M312 |
| TIME ALLOWED | : | THREE (3) HOURS |
| INSTRUCTIONS | : | 1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.<br>2. ANSWER ANY FIVE (5) QUESTIONS<br>3. ONLY NON-PROGRAMMABLE CALCULATORS MAY BE USED. |
| SPECIAL REQUIREMENTS | : | NONE |

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

## QUESTION 1

(a) In the natural coordinate system $(\overline{\tau}, \overline{n})$ derive the formulas for the

(i) velocity,

(ii) acceleration.

(b) Given $\overline{r}(t) = 2\cos 3t\overline{i} + 2\sin 3t\overline{j} + 3\overline{k}$. Find the

(i) trajectory,

(ii) velocity,

(iii) tangential acceleration,

(iv) normal acceleration,

(v) radius of curvature.

(c) A screw of a ship moving with a constant velocity 10m/s along a straight line has diameter 120cm and rotates in accordance with $\theta = 10\pi t$ rad ($t$ in seconds). Find

(i) a radius of curvature of the trajectory of a most distant point of a screw,

(ii) its velocity, and

(iii) acceleration.

(d) Prove that a spherical coordinate system is orthogonal.

[2,5,1,1,1,1,1,1,1,2,4]

## QUESTION 2

(a) Consider curvilinear coordinates $q_1, q_2, q_3$ in general case.

(i) Define the Lame parameters $H_i$.

(ii) Show that

$$H_i^2 = \left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2 .$$

(b) Let $\overline{e}_i$ and $\overline{e}_j$ be unit vectors corresponding to the coordinates $q_i$ and $q_j$ respectively.

(i) Find angles between $\overline{e}_i$ and the coordinate axes $OX, OY, OZ$.

(ii) Find the angle between $\overline{e}_i$ and $\overline{e}_j$, and hence

(iii) Derive a condition of orthogonality of $\overline{e}_i$ and $\overline{e}_j$. [2,2,4,3,2,3,4]

## QUESTION 3

(a) Consider the spherical coordinates as the special case of curvilinear, orthogonal ones. Find the formulas for

(i) velocity,

(ii) acceleration.

(b) Prove that grad $f$ is perpendicular to the surface $f = const.$

(c) Consider a function $f(x, y, z) = 4xz^3 - 3x^2y^2z$ and a point $\underline{P}(2, -1, 2)$.

(i) Find the directional derivative of $f$ at $\underline{P}$ in the direction of the vector $\overline{A}(2, -3, 6)$,

(ii) Compute the greatest rate of change of $f$ at $\underline{P}$.

(iii) Find the direction of the maximum rate of increase of $f$ at $\underline{P}$.

(d) Find the unit normal to the surface $z = x^2 + y^2$ at $\underline{P}(1, -1, 5)$.

[2,5,3,3,2,2,3]

## QUESTION 4

(a) Prove the following formulae

(i) $\nabla \times (\overline{A} \pm \overline{B}) = \nabla \times \overline{A} \pm \nabla \times \overline{B}$,

(ii) $\nabla \times (f\overline{A}) = f\nabla \times \overline{A} + \nabla f \times \overline{A}$,

(iii) $\nabla \cdot \overline{r} = 3$,

(iv) $\nabla \times \overline{r} = 0$.

(b) Give definition of the

(i) regular arc,

(ii) regular curve,

(iii) line integral of $\overline{U}$ over curve $C$.

(c) Determine the position vector of a particle whose velocity is

$$\overline{v} = e^t\overline{i} + e^{-t}\overline{j} + 3t\overline{k}, \quad t \geq 0.$$

and in the beginning a particle was at the origin. [3,4,3,3,1,1,2,3]

## QUESTION 5

(a) If $\overline{U} = y\overline{i} + x\overline{j}$, find the value of the line integral from $(0,0)$ to $(1,1)$

(i) along the parabola $y = x^2$,

(ii) along $y = 0$ from $x = 0$ to $x = 1$ and then along $x = 1$ from $y = 0$ to $y = 1$.

(b) Is $\overline{U}$ in (a) conservative? Explain.

(c) (i) State, and

(ii) Prove the necessary and sufficient condition theorem that a line integral in a simply connected region is independent of the path from point $A$ to point $B$.

[4,4,3,2,7]

## QUESTION 6

(a) Show that a

(i) necessary, and

(ii) sufficient condition that $\overline{U} = \nabla\phi$ is rot $\overline{U} = 0$.

(b) If $\overline{A} = 2yz\overline{i} - x^2y\overline{j} + xz^2\overline{k}$, and $\phi = 2x^2yz^3$ find

(i) $(\overline{A} \cdot \nabla)\phi$,

(ii) $\overline{A} \cdot \nabla\phi$,

(iii) $(\overline{A} \times \nabla)\phi$,

(iv) $\overline{A} \times \nabla\phi$.

[3,4,4,2,4,3]

## QUESTION 7

(a) (i) Apply Green's theorem to show that the area bounded by a simply closed curve $c$ is given by

$$\frac{1}{2}\int_c xdy - ydx,$$

(ii) and hence find the area of the ellipse $x = a\cos\theta, \quad y = a\sin\theta$.

(b) State;

(i) The divergence theorem,

(ii) Stokes's theorem.

(c) Prove that a necessary and sufficient condition that

$$\int_c \overline{A} \cdot d\overline{r} = 0$$

for every closed curve $c$ is that

$$\nabla \times \overline{A} = 0$$

identically. [4,4,3,3,6]