University of Swaziland

Final Examination – May 2009

BSc III, Bass III, BEd III

Title of Paper : Complex Analysis
Course Number : M313
Time Allowed : Three (3) hours
Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Express

$$\Gamma = \left(e^{-\frac{17}{4}\pi i} + \frac{1}{i}e^{\frac{61}{4}\pi i}\right)^2 + \left(i\cos\tfrac{\pi}{8} - \sin\tfrac{\pi}{8}\right)^{-4}$$

in the form $a + ib$. [8 marks]

(b) Find all values of z satisfying

$$\tanh z = -\tfrac{1}{2}.$$ [8 marks]

(c) Find the Taylor series expansion of $f(z) = \tan^{-1} z$ about $z = 0$. State the radius of convergence. [4 marks]

Question 2

(a) Express the equation

$$\mathrm{Re}\left(\frac{1}{\bar{z}}\right) = \tfrac{1}{2}$$

in terms of x and y. Hence describe and sketch the curve defined by this equation. [8 marks]

(b) Use the Theory of Residues to evaluate

$$\int_{-\infty}^{\infty} \frac{x^2\,\mathrm{d}x}{(x^2+4)^2}.$$ [12 marks]

Question 3

(a) Find the principal value of the complex number

$$\left[\tfrac{1}{2}(i-\sqrt{3})\right]^{2-\frac{1}{2}i}$$

and express it in the form $a+ib$. [8 marks]

(b) Consider the function $f(z)=\dfrac{1}{z^2\sin z}$.

i. Locate and classify all singularities of $f(z)$. [3 marks]

ii. Find the residue of $f(z)$ at each of the singular points. [9 marks]

Question 4

(a) Given the result

$$1+z+z^2+\cdots=\sum_{n=0}^{\infty}=\frac{1}{1-z},\quad |z|<1,$$

differentiate and find the sum of $\displaystyle\sum_{n=1}^{\infty} n^2z^n$. [14 marks]

(b) Show that

$$\cosh 2z=\cosh^2 z+\sinh^2 z.$$ [6 marks]

Question 5

(a) Evaluate

$$\oint_{|z|=1} \frac{z^2 e^{-2iz}}{(2z-i)^3} \mathrm{d}z. \qquad \text{[12 marks]}$$

(b) Find the derivative of

$$f(z) = \cot^{-1}\left(\frac{1-z}{1+z}\right)$$

and hence find the value(s) of z at which $f(z)$ is singular.

[8 marks]

Question 6

(a) Evaluate

$$\lim_{z\to 0}\Big(\cos z\Big)^{1/z^2}. \qquad \text{[6 marks]}$$

(b) Find two (2) Laurent series of

$$f(z) = \frac{1}{z(4z^2+1)}$$

stating the region of validity of each series. [6 marks]

(c) Evaluate

$$\int_{-1-i}^{2+2i} \left(2xy + ix^2\right) \mathrm{d}z$$

along the straight line joining the two points. [8 marks]

Question 7

(a) Write a short essay on

Laurent series and type classification of singularites.

[10 marks]

(b) Derive the formula

$$\tan^{-1} z = \frac{1}{2i} \ln\left(\frac{1+iz}{1-iz}\right).$$ [10 marks]