University of Swaziland

Supplementary Examination – July 2009

BSc III, Bass III, BEd III

Title of Paper : Complex Analysis

Course Number : M313

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Use *two methods* to evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2}. \qquad [16 \text{ marks}]$$

(b) A student remarked that

1 raised to any power always equals 1.

Is this statement true? Discuss. [4 marks]

Question 2

(a) Find the principal value of

$$\tanh z = i\sqrt{3}. \qquad [10 \text{ marks}]$$

(b) Consider the function

$$f(z) = \frac{1}{z^2(z^2+9)}.$$

i. Locate and classify all singularities of $f(z)$.
[2 marks]

ii. Find the residue of $f(z)$ at each of the singularities.
[8 marks]

Question 3

(a) Express the complex number

$$\left(-\tfrac{\sqrt{3}}{2}+\tfrac{i}{2}\right)^8+\tfrac{1}{4}(1-i\sqrt{3})^2$$

in the form $a+ib$. [10 marks]

(b) Integrate

$$\oint_{\Omega}\frac{\sin \pi z}{4z^2+1}\mathrm{d}z$$

where Ω is the circle $|z|=1$. [10 marks]

Question 4

(a) Give a full statement of the:

i. Cauchy's Integral Theorem [3 marks]

ii. Cauchy's Integral Formula [3 marks]

iii. Residue Theorem [3 marks]

(b) Show that if $f(z)=u(x,y)+iv(x,y)$ is an analytic function, then

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$ [6 marks]

(c) Determine whether the functions $u=e^{-x}(\cos y-\sin y)$ is harmonic. [5 marks]

Question 5

(a) Express each of the following equations in terms of x and y. Hence describe and sketch the curve defined by the equation.

i. $$\left|\frac{z-2i}{z+2i}\right|=1$$ [4 marks]

ii. $\left|\frac{z-2i}{z+2i}\right| = 2.$ [8 marks]

(b) Find the Taylor series of

$$f(z) = \tan z$$

about $z = \frac{1}{4}\pi$. Determine the region in which this series is convergent. [8 marks]

Question 6

(a) Use the Theory of Residues to evaluate

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}.$$ [12 marks]

(b) Prove that

$$|\sin z|^2 = \sin^2 x + \sinh^2 y.$$ [8 marks]

Question 7

(a) Analyse the convergence properties of the series

$$\sum_{n=0}^{\infty} \frac{(3+i)^n(z+2)^n}{n^2}.$$ [10 marks]

(b) Solve

$$z^4 + z^2 + 1 = 0$$

and express your answer(s) in the form $a + ib$.
[10 marks]