UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2008/9

BSc. /BEd. /B.A.S.S III

TITLE OF PAPER	:	ABSTRACT ALGEBRA I
COURSE NUMBER	:	M 323
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS. 3. Non-programmable calculators may be used.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Prove that every cyclic group is is abelian. [5 marks]

   (b) Let n be a positive integer greater than 1, and let $a, b \in \mathbb{Z}$

   $$aRb \text{ if and only if } a \equiv b (\text{mod } n)$$

   Prove that R is an equivalence relation on $\mathbb{Z}$. [7 marks]

   (c) Determine whether the set

   $$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}$$

   with respect to ordinary matrix multiplication gives a group structure.

   [8 marks]

QUESTION 2

2. (a) Prove that every finite group of prime order is cyclic. [5 marks]

   (b) Let $\phi : G \to H$ be an isomorphism of groups and e be the identity of the group G, then prove that

   (i) $(e)\phi$ is the identity of the group H. [4 marks]

   (ii) $(a^n)\phi = [(a)\phi]^n, \forall a \in G$ and $n \in \mathbb{Z}^+$. [8 marks]

   (c) Give the definition of a cyclic group. [3 marks]

QUESTION 3

3. (a) Prove that if $a \equiv b(\text{mod } n)$ and $c \equiv d(\text{mod } n)$, then $ac \equiv bd(\text{mod } n)$.

   [5 marks]

   (b) Prove that every subgroup of a cyclic group is cyclic. [10 marks]

   (c) Find the number of generators of cyclic groups of order 8 and 60. [5 marks]

QUESTION 4

4. (a) Solve the following system

$$2x \equiv 1 (\text{mod } 5)$$
$$3x \equiv 4 (\text{mod } 7)$$

[5 marks]

(b) Prove that if $a^2 = e, \forall a \in G$, then the group G is abelian. [3 marks]

(c) Using (4b) above show that a non-abelian group of order $2p$, p prime, contains at least one element of order p. [5 marks]

(d) Find the number of elements in each of the following cyclic subgroups.

(i) $< 30 >$ of $\mathbb{Z}_{42}$. [3 marks]

(ii) $< 15 >$ of $\mathbb{Z}_{48}$. [4 marks]

QUESTION 5

5. (a) Determine whether the set Q with respect to the binary operation

$$a \star b = a + b - 2009$$

is a group. [8 marks]

(b) Find the remainder when 7^{2009} is divided by 7. [5 marks]

(c) State the Lagrange's theorem. (Do not prove anything) [3 marks]

(d) Prove that if $(a, s) = 1$ and $(b, s) = 1$, then $(ab, s) = 1\, \forall a, b, s \in \mathbb{Z}$.

[4 marks]

QUESTION 6

6. (a) Prove that $\mathbb{Z}_p$ has no proper subgroups if p is a prime number. [6 marks]

   (b) Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 3 & 2 & 7 & 8 & 6 & 5 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 7 & 2 & 3 & 8 & 5 & 6 \end{pmatrix}$$

   (i) Express α and β as products of disjoint cycles and then as products of transpositions. For each permutation say whether it is an even permutation or an odd one. [8 marks]

   (ii) Compute $\alpha^{-1}, \beta^{-1}\alpha, (\alpha\beta)^{-1}$. [6 marks]

QUESTION 7

7. (a) Prove that if $d = (a, b)$, then there exist $x_0, y_0 \in \mathbb{Z}$ such that $d = x_0 a + y_0 b$. [6 marks]

   (b) For $\mathbb{Z}_{12}$, find all subgroups and give a lattice diagram. [7 marks]

   (c) Find the casets of $H = \{0, 6, 12\}$ in $\mathbb{Z}_{16}$. [4 marks]

   (d) Show that $\mathbb{Z}_6$ and $\mathbb{S}_3$ are not isomorphic. [3 marks]