UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION 2008/9

BSc. /BEd. /B.A.S.S III

TITLE OF PAPER	:	ABSTRACT ALGEBRA I
COURSE NUMBER	:	M 323
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS. 3. Non-programmable calculators may be used.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Give a single numerical example to disprove the following:
"If $ax \equiv bx (\text{mod } n)$ then $a \equiv b (\text{mod } n)$, $a, b, n \in \mathbb{Z}$." [8 marks]

(b) Find all subgroups of $\mathbb{Z}_{18}$ and the draw the lattice diagram. [8 marks]

(c) Let $\phi : G \to H$ be an isomorphism of groups and e be the identity of the group G, then prove that

(i) $(a^n)\phi = [(a)\phi]^n$. [4 marks]

(ii) $(e)\phi$ is the identity of the group H. [4 marks]

QUESTION 2

2. (a) (i) Define a normal subgroup N of a group G. [4 marks]

(ii) Show that the subgroup

$$N = \{(1), (123), (132)\}$$

is a normal subgroup of the group $G = S_3$. [6 marks]

(b) Prove that every subgroup of prime order is cyclic. [5 marks]

(c) Express $d = (2190, 465)$ as an integral linear combination of 2190 and 465. [5 marks]

QUESTION 3

3. (a) Consider the following permutations in S_6.

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$

Compute

(i) $\rho\sigma$ (ii) σ^2 (iii) σ^{-1} (iv) σ^{-2} (v) $\rho\sigma^2$ [10 marks]

(b) Write the permuations ρ and σ in (3a) as a product of disjoint cycles in S_6. [4 marks]

(c) Find the (a, b) and $[a, b]$ using the prime factorization ·

(i) $a = 216$, $b = 360$. [3 marks]

(ii) $a = 144$, $b = 625$. [3 marks]

QUESTION 4

4. (a) Solve the following

$$3x \equiv 5(\text{mod } 11)$$

[5 marks]

(b) Prove that if $(a,b)^{-1} = a^{-1}b^{-1}, \forall a, b \in G$, where G is a group, then G is abelian. [6 marks]

(c) Determine whether the set $G = \mathbb{Q} - \{0\}$ with respect to the operation

$$a \star b = \frac{ab}{10}$$

is a group. [9 marks]

QUESTION 5

5. (a) State Cayle's theorem (Do not prove anything). [4 marks]

(b) Consider $(\mathbb{R}^+, \cdot)$ and $(\mathbb{R}, +)$. Show that $(\mathbb{R}^+, \cdot)$ is isomorphic to $(\mathbb{R}, +)$. [6 marks]

(c) Find the number of generators in each of the following cyclic groups $\mathbb{Z}_{30}$ and $\mathbb{Z}_{42}$. [5 marks]

(d) Determine the right casets of $H = \{0, 4\}$ in $\mathbb{Z}_8$. [5 marks]

QUESTION 6

6. (a) For the binary operation $\star$ defined on a set G, say whether or not $\star$ gives a group structure on the set

(i) Define $\star$ on $\mathbb{R}$ by $a \star b = a + b - ab$. [6 marks]

(ii) $\star$-matrix multiplication and

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}$$ [8 marks]

(b) Show that $3\mathbb{Z}$ and $5\mathbb{Z}$ are isomorphic. [6 marks]

QUESTION 7

7. (a) Use Lagrange's theorem to show that $\mathbb{Z}_p$ has no proper subgroups. [5 marks]

(b) Show that if $(a, m) = 1$ and $(b, m) = 1$, then $(ab, m) = 1$, $a, b, m \in \mathbb{Z}$. [6 marks]

(c) (i) Find the conjugate elements of (12) in S_3. [4 marks]

(ii) Find the conjugate groups of $< (12) >$ in S_3. [5 marks]