UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2008/9

BSc. /BEd. /B.A.S.S III

TITLE OF PAPER	:	REAL ANALYSIS
COURSE NUMBER	:	M 331
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
		2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Let (x_n) be a sequence of real numbers and $l \in \mathbb{R}$. Explain precisely what is meant by the statement

$$\lim_{n\to\infty} x_n = l$$

Use this definition to show that

(i)

$$\lim_{n\to\infty} \frac{4n-9}{2n+5} = 2,$$

(ii) the sequence (x_n) defined by

$$x_n = \begin{cases} \frac{1}{3} & \text{if } n \text{ is divisible by } 3 \\ 0 & \text{otherwise} \end{cases}$$

does not converge to 0. [11 marks]

(b) Consider the sequence (x_n) defined by

$$x_1 = 2, \quad 7x_{n+1} = 2x_n^2 + 3, \text{ for } n \geq 1$$

(i) Show that $\frac{1}{2} < x_n < 3$ for all $n \geq 1$.

(ii) Show that (x_n) is a decreasing sequence.

(iii) Deduce that (x_n) is convergent and find its limit. [9 marks]

QUESTION 2

2. (a) Let S be a set of real numbers and $\gamma, \alpha, \beta \in \mathbb{R}$. Explain what is meant by

(i) S is bounded above.

(ii) γ is an upper bound for S.

(iii) α is the supremum for S.

(iv) β is the maximum of S. [8 marks]

(b) Find if they exist, the supremum and maximum for the following sets:

(i) $\{x \in \mathbb{R} : |x+5| \leq 1\}$

(ii) $\{x \in \mathbb{R} : 2x^2 - 5x - 3 > 0\}$

(iii) $\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$ [6 marks]

(c) Consider the statement; 'Every set of real numbers which is bounded above has a maximum'. Prove if true else give a counterexample. [2 marks]

(d) Let $S \subseteq \mathbb{R}$ be non-empty. Show that if $u = \sup S$, then $\forall n \in \mathbb{N}$, $u - \frac{1}{n}$ is not an upper bound of S but $u + \frac{1}{n}$ is an upper bound of S. [4 marks]

QUESTION 3

3. (a) Give an example of a function $f : [-1, 1] \to \mathbb{R}$ which is not continuous at $x = 0$. [3 marks]

(b) (i) State the Intermediate Value Theorem. [2 marks]

(ii) Use it to prove the following:

If $f : [a, b] \to [a, b]$ is continuous then $\exists x \in [a, b] : f(x) = x$. [6 marks]

The equation $\sin x + x = 1$ has a solution in the interval $[0, \frac{\pi}{2}]$. [3 marks]

(c) Determine whether each of the following statements is true or false giving a proof or a counterexample as appropriate.

(i) All continuous functions $f : [0, 1) \to \mathbb{R}$ attain a minimum value. [3 marks]

(ii) There is a continuous function $f : [0, 1) \to \mathbb{R}$ which is not bounded. [3 marks]

QUESTION 4

4. (a) Let $f : (a, b) \to \mathbb{R}$.

(i) Explain what is meant by the following:

f is continuous at $c \in (a, b)$. [2 marks]

f is differentiable at $c \in (a, b)$. [2 marks]

(ii) Prove that if f is differentiable at c then f is continuous at c. Give an example to show that the converse is false. [6 marks]

(b) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-x}, & x \leq 0 \\ 1 - x, & x > 0 \end{cases}$$

is continuous and differentiable everywhere and find its derivative $f'(x)$. [10 marks]

QUESTION 5

5. (a) (i) Prove by induction or otherwise that
$1^3 + 2^3 + \dots m^3 = \left[\frac{m(m+1)}{2}\right]^2, \forall m \in \mathbb{N}$ [4 marks]

(ii) Use this formula and the definition of the Riemann integral to show that
$$\int_0^1 x^3 dx = \frac{1}{4}$$
[10 marks]

(b) Determine whether each of the following statements is true or false. Prove or give counterexamples as appropriate.

(i) If neither $f : \mathbb{R} \to \mathbb{R}$ nor $g : \mathbb{R} \to \mathbb{R}$ is integrable then $f + g$ is not integrable. [3 marks]

(ii) A constant function is integrable. [3 marks]

QUESTION 6

6. (a) Given the series $\sum_{n=1}^{\infty} a_n$, define what is meant by the following:

(i) The n-th partial sum. [2 marks]

(ii) The sum. [2 marks]

(iii) Absolute convergence. [2 marks]

(iv) Conditional convergence. [2 marks]

(b) State and prove the squeeze theorem for sequences. [8 marks]

(c) Determine whether the following statement is true or false. Prove or give a counterexample as appropriate;
The series $\sum(-1)^n \frac{1}{n}$ is convergent. [4 marks]

QUESTION 7

7. (a) State the Mean Value Theorem. [2 marks]

(b) Use the Mean Value Theorem to prove the following statement;

Let $f : [a, b] \to \mathbb{R}$ be a function which is both continuous and differentiable on (a, b). If $f'(x) > 0 \; \forall x \in (a, b)$, then f is strictly increasing on (a, b).

[6 marks]

(c) Use the Mean Value Theorem to show that;

(i) $\frac{1}{7} < \sqrt{38} - 6 < \frac{1}{6}$. [4 marks]

(ii) $\frac{1}{2} < \ln 2 < 1$. [4 marks]

(iii) $\frac{b-a}{\sqrt{1-a^2}} < \cos^{-1} a - \cos^{-1} b < \frac{b-a}{\sqrt{1-b^2}}$ for $0 < a < b < 1$. [4 marks]