UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION 2008/9

BSc. /BEd. /B.A.S.S III

TITLE OF PAPER	:	REAL ANALYSIS
COURSE NUMBER	:	M 331
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Explain what is meant by 'the sequence of real numbers (x_n) converges to a limit $l \in \mathbb{R}$'.

   Use this definition to show that

   (i) If a sequence (x_n) converges to $l \in \mathbb{R}$ and a sequence (y_n) converges to $m \in \mathbb{R}$ then the sequence $(x_n + y_n)$ converges to $l + m$.

   (ii) the sequence $(x_n) = \left(\frac{\sin(\pi^2 n)}{n}\right)$ converges to 0.

   [11 marks]

   (b) Decide whether the following statements are true or false. Justify your answers.

   (i) There is a convergent sequence which is strictly decreasing.

   (ii) There is a sequence which is neither bounded below nor above.

   (iii) The sequence $(x_n) = \left(\sqrt{n+1} - \sqrt{n-1}\right)$ is convergent.

   [9 marks]

QUESTION 2

2. (a) Let S be a set of real numbers and $G, \gamma, \delta \in \mathbb{R}$. Explain what is meant by

   (i) S is bounded below.

   (ii) G is an lower bound for S.

   (iii) γ is the infimum for S.

   (iv) δ is the minimum of S. [8 marks]

   (b) Find if they exist, the infimum and minimum for the following sets:

   (i) $\left\{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\right\}$

   (ii) $\{x \in \mathbb{R} : 3x^2 + 2x - 8 < 0\}$

   (iii) $\{x \in \mathbb{R} : |x - 7| < 2\}$ [6 marks]

   (c) Consider the statement; 'If a set of real numbers has an infimum then it has a minimum'. Prove if true else give a counterexample. [2 marks]

   (d) Let $S \subseteq \mathbb{R}$ be non-empty. Show that if $u = \sup S$, then $\forall n \in \mathbb{N}$, $u - \frac{1}{n}$ is not an upper bound of S but $u + \frac{1}{n}$ is an upper bound of S. [4 marks]

QUESTION 3

3. (a) Given that,

$$f(x) := \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

determine whether the function f is continuous or not at $x = 0$. Justify your answer. [2 marks]

(b) Use the Intermediate Value Theorem to prove the following:

(i) If $f : [a, b] \to [a, b]$ is continuous then it has a fixed point in $[a, b]$. [6 marks]

(ii) The equation $\cos x = x$ has a solution in the interval $[0, \frac{\pi}{2}]$. [3 marks]

(c) Determine whether each of the following statements is true or false giving a proof or a counterexample as appropriate.

(i) All continuous functions $f : (0, 1] \to \mathbb{R}$ attain a maximum value. [3 marks]

(ii) All continuous functions $f : (0, 1] \to \mathbb{R}$ are bounded. [3 marks]

(iii) There is a function $f : [0, 1] \to \mathbb{R}$ that is discontinous at every point of $[0, 1]$ but such that $|f|$ is continuous on $[0, 1]$. [3 marks]

QUESTION 4

4. (a) Let $f : (a, b) \to \mathbb{R}$.

(i) Explain what is meant by saying that f is differentiable at $c \in (a, b)$. [2 marks]

(ii) Use this definition to show that:
$f(x) := |x - 1|$ is not differentiable at $x = 1$;
$f(x) := x^2 - 1$ is differentiable at every point $x = c$ with $f'(c) = 2c$. [8 marks]

(b) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} -e^x, & x > 0 \\ -1 - x, & x \leq 0 \end{cases}$$

is continuous and differentiable everywhere and find its derivative $f'(x)$. [10 marks]

QUESTION 5

5. (a) Let $f : [a,b] \to \mathbb{R}$. Use upper and lower sums to define the Riemann integral $\int_a^b f(x)dx$. [10 marks]

(b) From the definition of the Riemann integral show that

$$\int_0^1 x\,dx = \frac{1}{2}$$

Assume without proof that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \ \forall n \in \mathbb{N}.$$

[10 marks]

QUESTION 6

6. (a) Given the series $\sum_{n=1}^{\infty} a_n$, define the following.

(i) The n-th partial sum. [2 marks]

(ii) The convergence and the sum. [2 marks]

(b) State and prove the squeeze theorem for sequences. [8 marks]

(c) Consider the series $\sum_{n=2}^{\infty} \log\left(1 - \frac{1}{n^2}\right)$.

Show that the nth partial sum $s_n := a_2 + a_3 + \cdots + a_n$ is given explicity by $s_n = -\log 2 - \log n + \log(1+n)$. [8 marks]

QUESTION 7

7. (a) State the Mean Value Theorem. [2 marks]

(b) Use the Mean Value Theorem to prove the following statement;

Let $f : [a,b] \to \mathbb{R}$ be a function which is both continuous and differentiable on (a,b). If $f'(x) < 0 \ \forall x \in (a,b)$, then f is strictly decreasing on (a,b).

[6 marks]

(c) Use the Mean Value Theorem to show that;

(i) $\frac{1}{3} < \ln\frac{3}{2} < \frac{1}{2}$. [4 marks]

(ii) $\frac{1}{9} < \sqrt{66} - 8 < \frac{1}{8}$. [4 marks]

(iii) $\sin x < x$ for $0 < x < \pi$. [4 marks]