UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2008/9

TITLE OF PAPER : DYNAMICS II

COURSE NUMBER : M 355

TIME ALLOWED : THREE (3) HOURS

INSTRUCTIONS :

1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
2. ANSWER ANY FIVE QUESTIONS

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Prove that if the transformation equations are given by $r_\nu = r_\nu(q_1, q_2, \ldots, q_n, t)$ then

(i) $$\frac{\partial \dot{r}_\nu}{\partial \dot{q}_\alpha} = \frac{\partial r_\nu}{\partial q_\alpha}$$ (ii) $$\frac{d}{dt}\left(\frac{\partial r_\nu}{\partial q_\alpha}\right) = \frac{\partial \dot{r}_\nu}{\partial q_\alpha}.$$ [8]

(b) Define the Poisson bracket of the two quantities F and G. [3]

(c) If $f = f(p_\alpha, q_\alpha, t)$ and H is the Hamiltonian, prove that

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [H, f].$$

[4]

(d) For a certain dynamical system the kinetic and potential energy are given by

$$T = \frac{1}{2}\left((1+2k)\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2\right)$$
$$V = \frac{n^2}{2}\left((1+k)\theta^2 + \phi^2\right)$$

where θ and ϕ are generalized coordinates and n, k are positive constants. Write down Lagrange's equations of motion and deduce that

$$(\ddot{\theta} - \ddot{\phi}) + n^2\left(\frac{1+k}{k}\right)(\theta - \phi) = 0.$$

[10marks]

QUESTION 2

(a) A particle of mass m moves in one dimension such that it has the Lagrangian

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2 V(x) - V^2(x),$$

where V is some differentiable function of x. Show that the equation of motion reduces to

$$\left(m\ddot{x} + \frac{dV}{dx}\right)\left(m\dot{x}^2 + 2V(x)\right) = 0.$$

[10 marks]

(b) The kinetic and potential energy of a certain system are given by

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 \sin^2\alpha)$$
$$V = mgr\cos\alpha$$

where α is a constant. Use the Hamiltonian formulation to show that the angular momentum p_ϕ is conserved and is given by $p_\phi = mh\sin^2\alpha$ where $h = r^2\dot{\phi}$ is a known constant in the theory of forces. [10 marks]

QUESTION 3

(a) Use the Poisson bracket to show that the transformation

$$q = \sqrt{\frac{P}{\pi\omega}}\sin(2\pi Q) \ , \qquad p = \sqrt{\frac{\omega P}{\pi}}\cos(2\pi Q)$$

is canonical. [6 marks]

(b) Consider the transformation given by

$$Q = q^\alpha e^{\beta p} \ , \quad P = q^\alpha e^{-\beta p}$$

where α and β are constants. For which values of α and β is this transformation canonical? [5 marks]

(c) Given that,

$$\begin{aligned} A_1 &= \frac{1}{4}(x^2 + p_x^2 - y^2 - p_y^2), \quad A_2 = \frac{1}{2}(xy + p_x p_y) \\ A_3 &= \frac{1}{2}(xp_y - yp_x), \quad A_4 = x^2 + y^2 + p_x^2 + p_y^2 \end{aligned}$$

where x and y are generalized coordinates, and p_x and p_y are the generalized momenta, associates with x and y respectively. Evaluate

(i) $[A_1, A_2]$ [3 marks]

(ii) $[A_3, A_4]$ [3 marks]

(iii) $[A_3, A_2]$ [3 marks]

QUESTION 4

(a) Prove that if F has no explicit dependence on y then

$$\frac{\partial F}{\partial y'} = \text{Constant}$$

[3 marks]

(b) Prove that if F has no explicit dependence on y' then

$$F = \text{Constant}$$

[2 marks]

(c) Show that the Euler-Lagrange equation for the functional

$I = \int_a^b F(x, y, y', y'')dx$ is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

[15 marks]

QUESTION 5

Use three methods you have studied to check if the following transformation is canonical. State clearly which method you use each time.

$P = \frac{1}{2}(p^2 + q^2), \qquad Q = \arctan\left(\frac{q}{p}\right)$ [20 marks]

(*NB: do not use the poison bracket*)

QUESTION 6

(a) Find the extremals of the functional

$$\int_0^{\frac{\pi}{2}} (\dot{x_1}^2 + \dot{x_2}^2 + 2x_1x_2)dt$$

when $x_1(0) = 0, \quad x_1(\frac{\pi}{2}) = 1, \quad x_2(0) = 0, \quad x_2(\frac{\pi}{2}) = 1$ [9 marks]

(b) Consider a system with Hamiltonian

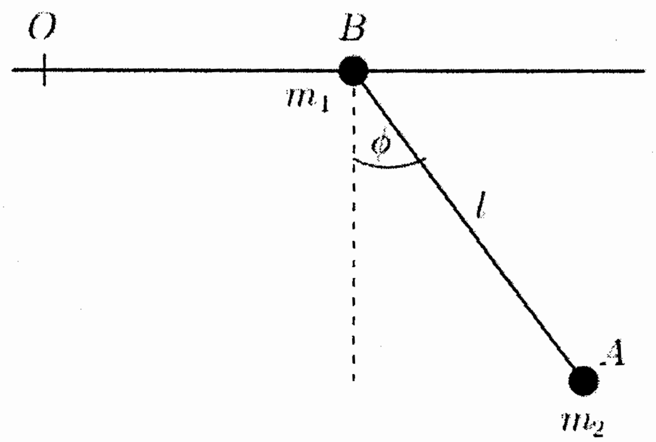
$$H = \frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2)$$

show that $M = p_1p_2 + q_1q_2$ and $L = p_1q_2 - q_1p_2$ are constants of motion by evaluating the poison brackets $[M, H]$ and $[L, H]$. [8 marks]

(c) Is $M - L$ a constant of motion? [3 marks]

QUESTION 7

A simple pendulum of mass m_2 is attached to a mass m_1 which can move freely along the horizontal line, as shown in the Figure below. The system is in a uniform gravitational field (acceleration g).



Choosing the generalized coordinates to be x, the distance moved by m_1 from O, and ϕ, the inclination of BA to the vertical,

(a) Write down the transformation equation. [5]

(b) Derive the Lagrangian function of the system. [6]

(c) Prove that Lagrange's equation of motion corresponding to the generalized coordinate ϕ is

$$l\ddot{\phi} + \ddot{x}\cos\phi + g\sin\phi = 0$$

[6]

(d) Write down the Lagrange's equation of motion corresponding to x. [3]