UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2008/9

BSc./ BEd./B.A.S.S IV

TITLE OF PAPER	:	NUMERICAL ANALYSIS II
COURSE NUMBER	:	M 411
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS. 3. CALCULATORS MAY BE USED.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Find the linear polynomial that best fits the data;

i	0	1	2	3
x_i	1	2	5	7
y_i	4	6	7	8

in the least squares sense. [10 marks]

(b) Approximate $x \ln x$ on $[1, 3]$ using a linear least squares polynomial. [10 marks]

QUESTION 2

2. (a) Use the Gram-Schmidt procedure to calculate $\phi_1(x)$ and $\phi_2(x)$ where $\{1, \phi_1(x), \phi_2(x)\}$ is an orthogonal set of polynomials on $[0, 1]$ with respect to the weight function $w(x) \equiv 1$. [12 marks]

(b) Hence (using your answer from part (a) above) for $f(x) \in C[0, 1]$, determine a and b such that $p(x) := a + b\phi_1(x)$ minimizes

$$\int_0^1 [f(x) - p(x)]^2 dx,$$

where $f(x) = x^2$. [8 marks]

QUESTION 3

3. (a) Discuss the consistency, zero-stability and convergence of the numerical scheme;

$$y_{k+1} = 3y_k - 2y_{k-1} + \frac{h}{2}[f_k - 3f_{k-1}]$$

for the Initial Value Problem (IVP):

$$y'(x) = f(x, y), \ a \leq x \leq b, \ y(a) = \alpha$$

[11 marks]

(b) Solve the IVP;

$$y'(x) = 3e^{-4x} - 2y, \ y(0) = 1$$

for $y(0.1)$ using one step of each of the following;

i. Euler method. [3 marks]

ii. Modified Euler method. [3 marks]

iii. Taylor series method of order 2. [3 marks]

QUESTION 4

4. Use a single step of the 4-th order Runge-Kutta method to solve the IVP:

$$x'' + 2x' + x = t,\ x(0) = 1, x'(0) = 3,$$

for $x(0.1)$ and $x'(0.1)$. [20 marks]

QUESTION 5

5. (a) Given the IVP;

$$y' = 1 + (t - y)^2,\ 2 \le t \le 3,\ y(2) = 1,$$

use the Taylor series method of order 2 with a step-size of $h = 0.1$ to approximate $y(2.2)$. [10 marks]

(b) Consider the differential problem;

$$\begin{aligned} u_{xx} + u_{yy} =& 0,\ 0 < x < 1,\ 0 < y < 1, \\ u(0, y) =& 0,\ u(1, y) = y,\ 0 \le y \le 1, \\ u(x, 0) =& 0,\ u(x, 1) = x,\ 0 \le y \le 1, \end{aligned}$$

Suppose the Finite Difference method is used to compute an approximate solution to this problem on a uniform grid with a stepsize of $h = \dfrac{1}{3}$. Determine (without solving) the resulting four difference equations in four unknowns. [10 marks]

QUESTION 6

6. Consider the differential problem;

$$\begin{aligned} u_t(x, t) =& u_{xx}(x, t),\ 0 < x < 1,\ t > 0, \\ u(0, t) =& u(1, t) = 0, t > 0, \\ u(x, 0) =& \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 1 - x, & \frac{1}{2} \le x \le 1 \end{cases} \end{aligned}$$

Suppose that a backward difference approximation and a central difference approximation are used for u_t and u_{xx} respectively. Then,

(a) Show that the resulting finite difference equations may be written in matrix form as

$$A\mathbf{u}^{(n)} = \mathbf{u}^{(n-1)},\ \text{where } n = 1, 2, \ldots$$

Identify the vector $\mathbf{u}^{(n)}$ and the square matrix A. [12 marks]

(b) Compute the leading terms of the truncation error for this numerical scheme. [8 marks]

QUESTION 7

7. Consider the hyperbolic differential equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad (1)$$

where $a > 0$ is a constant.

(a) Suppose a finite difference approximation is computed by using the explicit scheme

$$\frac{U_j^{n+1} - U_j^n}{k} + a\frac{U_j^n - U_{j-1}^n}{h} = 0$$

where U_j^n approximates $u(jh, nk)$ in the usual notation. Then, use the CFL condition to show that the given numerical scheme is convergent provided $a\frac{k}{h} \leq 1$. [10 marks]

(b) Determine the coefficients c_0, c_1 and c_{-1} so that the scheme,

$$U_j^{n+1} = c_{-1}U_{j-1}^n + c_0U_j^n + c_1U_{j+1}^n$$

for the solution of equation (1) agrees with the Taylor series expansion of $u(x_n, t_{n+1})$ to as high an order as possible when a is constant.

[10 marks]