University of Swaziland

Paper 2

BSc IV, Bass IV, BEd IV

Title of Paper : Partial Differential Equations
Course Number : M415
Time Allowed : Three (3) hours
Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.
5. A Table of Laplace Transforms is provided at the end of the question paper.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Consider the expression

$$u = x^2 f(y^2 - u^2) \qquad (1)$$

where $u = u(x, y)$ and f is an arbitrary function.
Find the partial differential equation for which (1) is a general solution. [10 marks]

(b) Use Laplace transforms to solve the system

$$\begin{aligned} u_{xx} - \frac{1}{c^2}u_{tt} &= -k \sin \pi x, \qquad 0 < x < 1,\ t > 0, \\ u(x,0) = u_t(x,0) &= 0, \qquad 0 \leq x \leq 1, \\ u(0,t) = u(1,t) &= 0, \qquad t \geq 0. \end{aligned}$$

[10 marks]

Question 2

Consider the equation

$$xu_{xx} + u_{yy} + u_x + u_y = x + 2.$$

a) Determine regions in which this equation is hyperbolic, parabolic or elliptic. [5 marks]

b) In each region, reduce the equation into its canonical form. [15 marks]

Question 3

Solve [20 marks]

$$\begin{aligned} u_t - u_{xx} &= 0, \qquad 0 < x < \pi,\ t > 0, \\ u(x,0) &= 2\cos x, \qquad 0 \leq x \leq \pi, \\ u_x(0,t) &= u(\pi,t) = 0, \qquad t \geq 0. \end{aligned}$$

Question 4

Solve the non-homogeneous boundary-value problem [20 marks]

$$\begin{aligned} u_t - u_{xx} &= e^{-t}\cos x, \qquad 0 < x < \pi,\ t > 0, \\ u(x,0) &= 2\sin^2 x, \qquad 0 \le x \le \pi, \\ u_x(0,t) &= u_x(\pi,t) = 0, \qquad t \ge 0. \end{aligned}$$

Question 5

Find the solution of the steady-state problem [20 marks]

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \qquad 0 < x < \pi,\ 0 < y < \pi, \\ u(0,y) &= y, \qquad 0 \le y \le \pi, \\ u(\pi,y) &= 0, \qquad 0 \le y \le \pi, \\ u(x,0) &= u(x,\pi) = 0,\ 0 \le x \le \pi. \end{aligned}$$

Question 6

(a) Find the particular solution of the PDE

$$yu_x - x^2u_y = xy$$

which contains the curve $u = x^2$ on $3y^2 = 2x^3$. [10 marks]

(b) Show that

$$\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 = \left(\frac{\partial F}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial F}{\partial \varphi}\right)^2$$

under the transformation $x = \rho\cos\varphi,\ y = \rho\sin\varphi$. [10 marks]

Question 7

Classify the equation

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$$

as hyperbolic, parabolic or elliptic. [5 marks]
Find the canonical form and hence the general solution of the equation. [15 marks]

Table of Laplace Transforms

$f(t)$	$F(s)$
t^n	$\frac{n}{s^{n+1}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n}{(s-a)^{n+1}}$
$\frac{1}{a-b}\left(e^{at}-e^{bt}\right)$	$\frac{1}{(s-a)(s-b)}$
$\frac{1}{a-b}\left(ae^{at}-be^{bt}\right)$	$\frac{s}{(s-a)(s-b)}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
$e^{at}\sin(at)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(at)$	$\frac{s-a}{(s-a)^2+b^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sin(at)\sinh(at)$	$\frac{2a^2}{s^4+4a^4}$
$\frac{\mathrm{d}^n f}{\mathrm{d}t^n}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$