University of Swaziland

Supplementary Examination – July 2009

BSc IV, Bass IV, BEd IV

Title of Paper : Partial Differential Equations
Course Number : M415
Time Allowed : Three (3) hours
Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Consider the expression

$$u = e^{xy} f(x + y - u) \qquad (1)$$

where $u = u(x, y)$ and f is an arbitrary function.
Find the partial differential equation for which (1) is a general solution. [10 marks]

(b) Solve the boundary-value problem

$$xu_{xx} + u_{xy} = 0, \quad u(1, y) = 1, \quad u_x(1, y) = e^y.$$

[10 marks]

Question 2

The electrostatic potential $u(r, \theta)$ inside a capacitor formed by two *hemi*-spheres insulated from each other and maintained at potentials 0 and V_0, respectively, obeys the system

$$\begin{aligned} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) &= 0, \qquad 0 < r < a,\ 0 < \theta < \pi \\ u(a, \theta) &= \begin{cases} V_0, & 0 < \theta < \frac{1}{2}\pi \\ 0, & \frac{1}{2}\pi < \theta < \pi \end{cases} \end{aligned}$$

Solve for $u(r, \theta)$ inside the capacitor. [20 marks]

Question 3

Solve [20 marks]

$$\begin{aligned} u_t - u_{xx} &= 0, \qquad 0 < x < \pi,\ t > 0, \\ u(x, 0) &= 2\cos x, \qquad 0 \le x \le \pi, \\ u(0, t) &= u(\pi, t) = 0, \qquad t \ge 0. \end{aligned}$$

Question 4

Consider the equation

$$y^2u_{xx} - 2yu_{xy} + u_{yy} = u_x + 6y.$$

By first reducing it into its canonical form, find the general solution of this equation. [20 marks]

Question 5

Consider the Cauchy problem for the wave equation

$$\begin{aligned} u_{tt} &= c^2u_{xx}, & -\infty < x < \infty,\ t \geq 0, \\ u(x,0) &= f(x), & -\infty < x < \infty, \\ u_t(x,0) &= g(x), & -\infty < x < \infty. \end{aligned}$$

Derive the d'Alembert's solution

$$u(x,t) = \frac{1}{2}\Big\{f(x+ct) + g(x-ct)\Big\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\alpha)\mathrm{d}\alpha.$$

[20 marks]

Question 6

(a) Find the particular solution of the PDE

$$xu_x + yu_y = u$$

which contains the curve $x + y = 1, \quad uy = 1.$ [10 marks]

(b) Find an expression of the Laplacian

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}$$

in terms of u and v, where $x = u\cos\alpha - v\sin\alpha, \quad y = u\sin\alpha + v\cos\alpha.$ [10 marks]

Question 7

(a) Use Laplace transforms to solve the bondary-value problem [10 marks]

$$\begin{aligned} u_{xt} + \sin t &= 0, \quad -\infty < x < \infty, \quad t \geq 0, \\ u(x,0) &= x, \quad -\infty < x < \infty, \\ u(0,t) &= e^{-t}, \quad t \geq 0. \end{aligned}$$

(b) Use any other method to solve the problem in (a). [10 marks]