UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2008/09

BSc.IV

TITLE OF PAPER	:	METRIC SPACES
COURSE NUMBER	:	M431
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE (5) QUESTIONS 3. ONLY NON-PROGRAMMABLE CALCULATORS MAY BE USED.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY TEH INVIGILATOR.

QUESTION 1

(a) Consider the process $f : X \to X, \quad x \in X, \ \ y = f(x)$, where

$$f((x)(t)) = y(t) = 1 + \int_o^t u^2 x(u) du.$$

(i) Find at least 4 terms in the iterative process

$$x_{n+1} = f(x_n), \quad x_1 = 1.$$

(ii) Find the fixed point, thus solve the integral equation

$$x(t) = 1 + \int_0^t u^2 x(u) du.$$

[4,3]

(b) (i) Show that metric space axioms hold for L_1-metric.

(ii) Find distance between $\sin t$ and $\cos t, \quad t \in [0, 2\pi]$ in L_1-metric. [3,4]

(c) Let a sequence in $C[-1, 1]$ be given by

$$x_n(t) = \begin{cases} 0 & \text{if } -1 \leq t \leq 0 \\ nt & \text{if } 0 < t < \frac{t}{n} \\ 1 & \text{if } \frac{1}{n} \leq t \leq 1 \end{cases}$$

Show that this is a Cauchy sequence in L_1-metric. [6]

QUESTION 2

(a) (i) Define a metric space.

(ii) Show that $d(x, y) \geq 0$ for any x and y in a metric space. [2,2]

(b) Let X be the set of continuous functions $f : [a, b] \to R$. Show that X together with

(i) Max-metric,

(ii) Sup-metric,

is a metric space. [3,3]

(c) Evaluate the distance between $\sin t$ and $\cos t$, $\quad t \in [0, 2\pi]$

using the max-metric. [4]

(d) (i) Define a disconnected set in a metric space (X, d).

(ii) Apply the above definition to show that the set F is disconnected, where

$F\{(x,y): \; x^2 + y^2 \leq 1\}\, U\, \{(x,y): \; x^2 + y^2 \geq 2\}$. [3,3]

QUESTION 3

(a) Let $X = R^2$. Show that X together with Euclidean distance forms a metric space.

[5]

(b) Prove that uniform convergence is stronger than the pointwise convergence. [4]

(c) (i) Let $f_n(x) = \frac{1}{nx}$, for $x > 0$ and $n \in N$. Show that $\{f_n\}$ converges pointwise on $X = (0, \infty)$, but not uniformly.

(ii) Let $f_n(x) = \frac{1}{nx}$, for $x \geq a$ and $n \in N$. Show that

$f_n \to 0$ uniformly on $X = [a, \infty)$. [3,3]

(d) Apply M-test to verify the statements in (c). [5]

QUESTION 4

(a) Consider an arbitrary non-empty set X together with the discrete metric, $x, y \in X$

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2 & \text{if } x \neq y \end{cases}$$

Show that (X, d) is a metric space. [4]

(b) Let (X, d) be a metric space. Give the definition of

(i) a convergent sequence in (X, d),

(ii) a closed set $A \subset X$. [2,2]

(c) Prove that any intersection of closed sets in a metric space is itself closed. [5]

(d) Let $F : R^2 \to R$ be continuous and let A be closed in R. Prove that $F^{-1}(A)$ is closed in R^2. [4]

(e) Consider the sequence $(0,1)$, $(\frac{1}{2}, 1\frac{1}{2})$, $(\frac{2}{3}, 1\frac{2}{3})$, $(\frac{3}{4}, 1\frac{3}{4}), \cdots$ to illustrate that the set

$A = \{(x,y) : \quad x^2 + y^2 < 5\}$ is not closed. [3]

QUESTION 5

(a) Show that $X = R^2$ together with the New York metric forms a metric space. [6]

(b) Prove that any contraction is uniformly continuous. [4]

(c)Show that the mapping

$F : [-1, 1] \to [-1, 1]$ defined by

$$f(x) = \frac{1}{8}(x^3 + 2x^2 + 4)$$

is a contraction and deduce that there is unique solution to the equation $x^3 + 2x^2 - 8x + 4 = 0$ in the interval $[-1, 1]$. [6]

(d) Show that the function $f : R \to R$ given by $f(x) = \cos x$ is not a contraction, but the function $f(x) = \frac{9}{10}\cos x$ is a construction. [4]

Hint: Apply a derivative test.

QUESTION 6

(a) (i) Define the Chicago distance.

(ii) Show that $X = R^2$ together with a Chicago distance form a metric space. [2,3]

(b) In a metric space (X, d) define

(i) a Cauchy sequence

(ii) a complete set $A \subset X$,

(iii) a compact set $A \subset X$. [2,2,2]

(c)Prove that any convergent sequence in a metric space in a Cauchy sequence. [4]

(d) Take $X = (0, \infty)$ and $d(x, y) = |x - y|$.

(i) Show that $x_n = \frac{1}{n}$, $n \in N$ is a Cauchy sequence in (X, d) but not convergent in X.

(ii) Show that (X, d) is not a complete metric space. [3,2]

QUESTION 7

(a) Show that

(i) (R^n, dp), and

(ii) Lp

metric spaces formally satisfy the metric space axioms. [5,5]

Hint: You may use the Minkowski's inequalities for the complex numbers and for the continuous functions.

(b) Given an open ball $B_\varepsilon(x)$ in a metric space and a point y in $B_\varepsilon(x)$. Prove that there exists $\delta > 0$ such that $B_\delta(y) \subset B_\varepsilon(x)$. [5]

(c) Prove that if A is a compact set in a metric space then A is complete. [5]

END OF EXAMINATION