UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION 2008/09

BSc.IV

TITLE OF PAPER : METRIC SPACES

COURSE NUMBER : M431

TIME ALLOWED : THREE (3) HOURS

INSTRUCTIONS :
1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
2. ANSWER ANY FIVE (5) QUESTIONS
3. ONLY NON-PROGRAMMABLE CALCULATORS MAY BE USED.

SPECIAL REQUIREMENTS : NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY TEH INVIGILATOR.

QUESTION 1

(a) (i) Define a metric space.

(ii) Show that for any $x, y, z_1, \cdots z_n$ from a metric space

$$d(x,y) \leq d(x,z_1) + d(z_1,z_2) + \cdots + d(z_{n-1},z_n) + d(z_n,y).$$

2,2]

(b) Let $X = C$ be a set of complex numbers, and $d(x,y) = |x-y|$.

(i) Show that for any $x, y \in X$

$$|x+y| \leq |x| + |y|.$$

(ii) Thus prove that (X,d) is a metric space. [3,3]

(c) Evaluate the distanc between $x = t^2$ and $x = t+1, \quad t \in [0,2]$ in max-metric.

[4]

(d) (i) Define a disconnected set in a metric space (X,d).

(ii) Apply the above definition to show that the set H is disconnected, where

$$H = \left\{(x,y) \;\; x^2+y^2 \leq 2\right\} \;\; \cup \left\{(x,y): \;\; x^2+y^2 \geq 2.5\right\}$$

[3,3]

QUESTION 2

(a) Consider the process $f : X \to X, \quad x \in X, \quad y = f(x)$, where

$$y(t) = 1 = -\int_0^t [x(u)]^2 du, \quad t \in (-1, 1)$$

(i) Find at least 4 terms in the iterative process

$$x_{n+1} = f(x_n), \quad x_1 = 0.$$

(ii) Show that $x(t) = \frac{1}{1+t}$ is a fixed point, thus solve the integral equation

$$x(t) = 1 - \int_0^t x^2 du.$$

[4,3]

(b) Show that metric space axioms hold for L_1−metric. [3]

(c) Find distance between t^2 and $t+1, \quad t \in [0, 2]$ in L_1-metric. [4]

(d) Let a sequence in $C[-1, 1]$ be given by

$$x_n(t) = \begin{cases} 0 & \text{if } -1 \le t \le 0 \\ nt & \text{if } 0 < t < \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} \le t \le 1 \end{cases}$$

Prove that this is a Cauchy sequence in L_1-metric. [6]

QUESTION 3

(a) Let $X = R^2$. Show that X together with the lift distance is a metric space. [4]

(b) Prove that uniform convergence is stronger than the pointwise convergence. [4]

(c)(i) Put the negation of the difiniton of uniform convergence.

(ii) Thus show that the sequence $f_n(x) = x^n, \quad x \in [0,1]$ converges pointwise but not uniformly on $[0,1]$. [3,3]

(d) (i) State and

(ii) Prove M-test for uniform convergence. [2,4]

QUESTION 4

(a) Show that an arbitrary non-empty set X together with the distance d defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 3 & \text{if } x \neq y \end{cases}$$

for all $x, y \in X$

forms a metric space. [4]

(b) Let (X, d) be a metric space. Give a definition of

(i) Convergent sequence in (X, d).

(ii) Open set $A \subset X$ [2,2]

(c) Let $A_1, A_2, \cdots, A_k$ be closed sets in a metric space. Prove that $A_1 \bigcup A_2 \bigcup \cdots \bigcup A_k$ is closed. [6]

(d) Define in a metric space

(i) an open ball,

(ii) continuous map $f : M_1 \rightarrow M_2$ between metric spaces using open balls notion.

[3,3]

QUESTION 5

(a) Show that $X = R^2$ together with the London metric forms a metric space. [6]

(b) Prove that any contraction is uniformly continuous. [4]

(c) Show that the mapping

$f : [0,1] \to [0,1]$ defined by

$$f(x) = \frac{1}{7}(x^3 + x^2 + 1)$$

is a contraction and deduce that there is a unique solution to the equation

$x^3 + x^2 - 7x + 1 = 0$ in the interval $[0,1]$. [6]

(d) Confirm by differentiation that the function $f : [1,\infty) \to [1,\infty)$ given by $f(x) = x + \frac{1}{x}$ is not a contraction, but that the function $g : [1,\infty) \to [1,\infty)$ given by $g(x) = \frac{9}{10}(x + \frac{1}{x})$ is. [4]

QUESTION 6

(a) (i) Define the New York distance.

(ii) Show that $X = R^2$ together with the New York distance form a metric space [2,3]

(b) In a metric space (X, d) define

(i) a Cauchy sequence,

(ii) a complete set $A \subset X$,

(iii) a compact set $A \subset X$. [2,2,2]

(c) Give an example illustrating that there is a Cauchy sequence in a metric space (X, d) which is not convergent in X. [4]

(d) Let $\{x_n\}$ be a Cauchy sequence in a metric space (X, d) and let $\{x_{n_k}\}$ be a convergent subsequencce of $\{x_n\}$. Prove that $\{x_n\}$ is convergent itself. [5]