# UNIVERSITY OF SWAZILAND

## FINAL EXAMINATION 2008/09

## BSc.VI

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| TITLE OF PAPER | : | FLUID DYNAMICS |
| COURSE NUMBER | : | M455 |
| TIME ALLOWED | : | THREE (3) HOURS |
| INSTRUCTIONS | : | 1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.<br>2. ANSWER ANY FIVE (5) QUESTIONS<br>3. ONLY NON-PROGRAMMABLE CALCULATORS MAY BE USED. |
| SPECIAL REQUIREMENTS | : | NONE |

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

## USEFUL FORMULAE

The gradient of a function $\psi(r, \theta, z)$ is cylindrical polar coordinates is

$$\nabla\psi = \frac{\partial\psi}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial\psi}{\partial r}\underline{\hat{\theta}} + \frac{\partial\psi}{\partial r}\underline{\hat{k}}.$$

The divergence and curl of a vector field

$$\underline{v} = v_r\underline{\hat{r}} + v_\theta\underline{\hat{\theta}} + v_z\underline{\hat{k}}$$

in cylindrical polar coordinates are

$$\nabla \cdot \underline{v} = \frac{1}{r}\left\{\frac{\partial}{\partial r}(rv_r) + \frac{\partial}{\partial\theta}(v_\theta) + \frac{\partial}{\partial z}(rv_z)\right\},$$

and

$$\nabla \times \underline{v} = \frac{1}{r} det \begin{bmatrix} \underline{\hat{r}} & r\underline{\hat{\theta}} & \underline{\hat{k}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ v_r & rv_\theta & v_z \end{bmatrix}.$$

The divergence of a vector

$$\underline{v} = v_r\underline{\hat{r}} + v_\lambda\underline{\hat{\lambda}} + v_\theta\underline{\hat{\theta}}.$$

is spherical coordinates

$$\nabla \cdot \underline{v} = \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v_\lambda}{\partial\lambda} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta v_\theta)}{\partial\theta}.$$

The convective derivative, Laplacian and strain and shear stress in cylindrical coordinates are:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_r\frac{\partial}{\partial_r} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + v_z\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}.$$

$$e_{r\theta} = \frac{1}{2}r\frac{\partial}{\partial r}(\frac{V_0}{r}) + \frac{1}{2}\frac{1}{r}\frac{\partial V_r}{\partial\theta}, \quad s_{r\theta} = 2\mu e_{r\theta}$$

Identities:

$$\underline{V} \cdot \nabla \underline{V} = \nabla(\frac{1}{2}V^2) - \underline{V} \times \underline{\omega}.$$

$$\nabla \times (\nabla \times \overline{A}) = \nabla\nabla \cdot \overline{A} - \nabla^2 \overline{A}.$$

## QUESTION 1

(a) Consider steady, one-dimensional, incompressible flow along the x-axis through the converging channel

$$\overline{v} = v_1(1 + \frac{x}{L})\overline{i},$$

where $v_1$ and $L$ are constants.

(i) Find the acceleration of a particle moving along the x-axis as a function of $x$.

(ii) For the particle located at $x = 0$ at $t = 0$, obtain an expression for its

-position, $x_p$ as a function of time for $t > 0$,

-$x$ component of acceleration $a_{xp}$, as a function of time for $t > 0$.

(iii) Find the Eulerian acceleration for particle moving along x-axis.

(b) For the three-dimensional flow

$$\overline{v} = (xt, -y, b(t))$$

find the

(i) particle paths,

(ii) streamlines.

[4,3,2,1,5,5]

## QUESTION 2

(a) Derive the continuity, or mass conservation equation for a fluid.

(b) Write down the equation in (a) for a steady flow.

(c) For a steady incompressible two-dimensional flow in the $xy-$plane the $x$ component of velocity is given by $u = ax$, where $a$ is a constant.

(i) Derive a possible $y$ compoent.

(ii) How many possible $y$ components are there?

(d) Let $\overline{v} = mr^{-2}\hat{r}$ in spherical coordinates.

(i) Show that $\overline{v}$ satisfies the continuity equation for incompressible flow, except at the origin 0.

(ii) Let 0 lye on a smooth surface $S$. What is the volume flow rate through $S$?

[5,1,3,1,5,5]

## QUESTION 3

(a) Describe two methods of modelling continuous medium.

(b) Describe a continuum model of a fluid.

(c) Derive a formula for the convective derivative of the density.

(d) Consider the stream function

$$\psi(r,\theta) = rV(\theta\cos\theta + \frac{1}{2}\pi\theta\sin\theta - \sin\theta)$$

for $0 \leq \theta \leq \frac{1}{2}\pi$, $\quad V$ is a constant.

(i) Calculate the velocity components on $\theta = 0$ and $\theta = \frac{1}{2}\pi$.

(ii) Calculate $\nabla^2\psi$.

[4,4,5,4,3]

## QUESTION 4

(a) Consider the two-dimensional flow field given by $\psi = ax^2 - ay^2$, $a$ is a constant

Show that the flow is irrotational.

(b) Calculate the vorticity and the angular velocity (rotation) for the following flows:

(i) "rigid rotation": $v_\theta = Dr, v_r = 0$;

(ii) "line vortex": $v_\theta = \frac{c}{r}, v_r = 0$;

(iii) "uniform shear flow": $u = \beta y, v = 0$.

(c) Give the definitions of a vortex line, a vortex surface, a vortex tube, and strength of a vortex tube.

(d) State Kelvin's theorem. [6,2,2,2,4,4]

## QUESTION 5

(a) Give the definition of a Newtonian fluid.

(b) Use (a) to find the dimensions of

(i) Viscosity,

(ii) Kinematic Viscosity.

(c) Derive the Navier-Stokes equation in the form

$$\rho \frac{DVi}{Dt} = \rho Fi + \frac{\partial}{\partial x_j}\sigma_{ij}, \quad i = 1, 2, 3.$$

(d) The plane $y = 0$ oscillates so that its velocity is in the plane $y = 0$ and magnitude $V \cos \omega t$, $(V, \omega$ are constants).

Find the velocity of viscous impressible flow above the plane. Body forces are negligible. [2,2,2,7,7]

## QUESTION 6

(a) (i) Give the definition of Reynold's number.

(ii) Show that Reynolds number is dimsionless.

(b) (i) Define simila flows.

(ii) How is the idea of the similarity of flows used in the design of experimental models?

(c) Consider a cylinder of radius $a$ rotating with angular velocity $\Omega$ in a viscous incompressible fluid.

(i) Show that the steady cirucar flow inside the cylinder is in rigid body motion, and outside the cylinder is that associated with a line vortex.

(ii) Calculate the moment of stress

$$\int_0^{2\pi} S_{r\theta} \ a \ d\theta$$

on inner and outer surfaces of the cylinder. [2,2,2,2,7,5]

## QUESTION 7

(a) Consider steady incompressible invicid potential flow.

(i) Show that

$$\overline{v} \times \overline{\omega} = \nabla[\frac{1}{2}\overline{v}^2 + \Phi + \frac{p}{\rho}].$$

(ii) Derive Bernoulli's equation.

(b) An infinite plane starts to move at speed $U_0$ in its own plane at $t = 0$. The viscous incompressible fluid is initially at rest. The effect of body forces is negligible. Put $\overline{v} = U(y,t)\overline{i}$.

(i) State the initial-bounday problem for $U(y,t)$,

(ii) Solve the problem in (i) by introducing a new dimensionless variable $\xi = y/(4\nu t)^{\frac{1}{2}}$, and using $U(y,t) = U_0 f(\xi)$. [4,4,4,8]

END OF EXAMINATION