UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2009/10

BSc./B.Ed./B.A.S.S II

TITLE OF PAPER	:	LINEAR ALGEBRA
COURSE NUMBER	:	M220
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

a) Let V be all ordered pairs of real numbers. Define addition and scalar multiplication as follows

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y + y_2 + 1)$ and $\alpha(x_1, y_1) = (\alpha x_1 + \alpha - 1, \alpha y_1 + \alpha - 1)$.

Show that V is a vector space

[10]

(b)Give that $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{pmatrix}$. Use the augmented matrix $[A : I]$ to compute A^{-1}.

[5]

(c) Determine whether the following has a non-trivial solution:

$$\begin{aligned} 2x + y - z + 2w &= 0 \\ x + y + z + w &= 0 \\ 3x + 2y + 2z + 2w &= 0 \end{aligned}$$

[5]

QUESTION 2

(a) Use Crammer's rule to solve(i) and use Gaussian elimination to solve (ii)

(i)

$$\begin{aligned} 2x + 2y + z &= 1 \\ 3x + y + z &= 2 \\ x + y + z &= 0 \end{aligned}$$

(ii)

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ 2x_1 + 3x_2 + x_3 &= 5 \\ x_1 + x_2 - 2x_3 &= -5 \end{aligned}$$

[10]

(b) Find the inverses by inspection

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

[5]

(c) Given that $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$. Verify Cayley-Hamilton theorem for the matrix A.

[5]

QUESTION 3

(a) Determine whether the sets of vectors in the vector space V are linearly dependent or independent

i $\{2x^2 + x, x^2 + 3, x\}$ $\quad V = P_2(x)$

ii $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ $\quad V = M_2(\mathbb{R})$

[10]

(b) Let $s = \{\nu_1, \nu_2, \cdots, \nu_n\}$ be a set of non-zero vectors in a vector space V. Prove that s is linearly dependent $\Leftrightarrow$ one of the vectors is a linear combination of the preceeding vectors in s.

[10]

QUESTION 4

(a) Find conditions on λ and μ for which the following system has

i a unique solution

ii no solution or

iii infinitely many solutions.

$$\begin{aligned} x + y - 4z &= 0 \\ 2x + 3y + z &= 1 \\ 4x + 4y + \lambda z &= \mu \end{aligned}$$

[10]

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $T(x, y) = (x - 2y, 2x + y, x + y)$

i Find the standard matrix of T

ii Find the matrix of T with respect to B^1 and B where $B^1 = \{(1, -1), (0, 1)\}$ and $B = \{(1, 1, 0), (0, 1, 1), (1, -1, 1)\}$

[10]

QUESTION 5

(a) Prove that if a homogeneous system has more unknowns than the number of equations then it has a non-trivial solution.

[10]

(b) Find the characteristic polynomial eigenvalues and eigenvectors of the following matrix.

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & -2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

[10]

QUESTION 6

(a) Show that B is a basis for $\mathbb{R}^3$, where

$$B = \{(0, 2, 1), (1, 0, 2), (1, -1, 0)\}$$

[5]

(b) Show that the vector $\begin{pmatrix} 12 \\ 12 \\ -3 \end{pmatrix}$ is a linear combination of the vectors $(2, 0, 1)^T, (4, 2, 0)^T, (1, 3, -1)^T$

[5]

(c) Let $B_1 = \left\{ \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$ and

$$B_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$ be bases. Find the transition matrix from B_1 to B_2

[10]

QUESTION 7

(a) Given that

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & 5 & -3 \\ 0 & -1 & 3 & 0 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Use the augemented matrix $[A : I]$ to find A^{-1}.

(b)In (a) above find a finite sequence of elementary matrices $E_1, E_2, \cdots, E_k$ such that $E_k E_{k-1} \cdots E_1 A = I$.

[13]

(c) Evaluate the following determinant using cofactor expansion along the second row

$$\begin{vmatrix} 3 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -3 & 2 \end{vmatrix}$$

[4]

(d) Determine whether the system has a nontrivial solution

$$\begin{aligned} -x_1 + 2x_2 + 2x_3 + 2x_4 &= 0 \\ 3x_1 + x_2 - x_3 + 2x_4 &= 0 \\ x_1 - 2x_2 + 3x_3 - x_4 &= 0 \end{aligned}$$

[3]