UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2009/2010

B.Sc. / B.Ed. / B.A.S.S. II

TITLE OF PAPER	:	FOUNDATIONS OF MATHEMATICS
COURSE NUMBER	:	M231
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Write down the negation of the following statement: "The function f of one variable is a *convex function* if and only if for all real numbers x and y and for all real numbers t with $0 \leq t \leq 1$, it follows that $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$." [6]

(b) Which of the following statements are true?

(i) Some animals are four legged, whereas all the rest are two legged. [1]

(ii) The square root of any integer is a non-negative real number. [1]

(c) For each of the following, write the *converse* and the *contrapositive*:

(i) If n is an integer for which n^2 is even, then n is even. [3]

(ii) Suppose that t is an angle between 0 and π. If t satisfies $\sin(t) = \cos(t)$, then $t = \frac{\pi}{4}$. [3]

(d) Prove that the following statements are false:

(i) For all $n \in \mathbb{N}$, $n^2 - n + 87$ is a prime number. [2]

(ii) For all $n \in \mathbb{N}$, $2n^2$ is an odd integer. [2]

(iii) For some $n \in \mathbb{N}$, with $n \geq 2$, $n^2 + 2n$ is a prime integer. [2]

QUESTION 2

(a) What do you understand by the following?

(i) Premiss; [1]

(ii) Deductive reasoning; [1]

(iii) Inductive reasoning; [1]

(iv) Mathematical Proof. [1]

(b) Using the axioms given below, prove each of the theorems which follow.

Axiom 1 All mathematicians are logical.

Axiom 2 Careful people are not foolish.

Axiom 3 Discontented people are foolish.

Axiom 4 Logical people are careful.

Theorem 1 *Mathematicians are contented.*

Theorem 2 *Foolish people are not logical.*

Theorem 3 *Careless people are not mathematicians.*

[6]

(c) Consider the proposition "If z is a real number that satisfies $z^3+3z^2-9z-27 \geq 0$, then $|z| \geq 3$."

(i) Reword the proposition so that it is in the form "A implies C OR D." [2]

(ii) Prove the proposition by assuming that A and $NOT\, C$ are true. [4]

(iii) Prove the proposition by assuming that A and $NOT\, D$ are true. [4]

QUESTION 3

(a) Give the definition of a *valid argument*. [1]

(b) Using truth tables, analyze the following argument and then state whether it is valid or invalid.

> "It is not true that he is rich and arrogant. He is rich. Therefore he is not arrogant."

[6]

(c) Prove that if there are at least 6 people at a party, then either 3 of them knew each other before the party, or 3 of them were complete strangers before the party. [9]

(d) Show that the polynomial $p(x) = x^4 - 2x^2 - 3$ has a root that lies between $x = 1$ and $x = 2$. [4]

QUESTION 4

(a) Suppose that $a_0.a_1a_2a_3\dots$ and $b_0.b_1b_2b_3\dots$ are two different decimal representations of the same real number. Prove that one of these expressions ends in $9999\dots$ and the other in $0000\dots$. [8]

(b) Prove that between any two distinct real numbers there is a rational number and an irrational number. [12]

QUESTION 5

(a) State and prove the Principle of Strong Mathematical Induction. [6]

(b) The *Fibonacci sequence* is a sequence of integers $u_1, u_2, \ldots, u_n, u_{n+1}, \ldots$, such that $u_1 = 1$, $u_2 = 1$ and

$$u_{n+1} = u_n + u_{n-1}$$

for all $n \geq 2$. The beginning of this sequence is $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$. Prove by strong induction, or by any other method, that for all positive integers n,

$$u_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n),$$

$$\text{where} \quad \alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}$$

. [8]

(c) Suppose that Canada Post prints only 3 cent and 5 cent stamps. Prove that it is possible to make up any postage of n cents using only 3 cent and 5 cent stamps for $n \geq 8$. [6]

QUESTION 6

(a) Give the definition of a countable set. [3]

(b) Show that if A and B are countable sets, then $A \cup B$ is also countable. [9]

(c) Prove that the set of all rational numbers is countable. [8]

QUESTION 7

(a) Let x be a rational number. Prove that there exists a monotone decreasing sequence $(x_n)_{n \geq 0}$ of irrational numbers which converges to x. [5]

(b) Let X and Y be any sets, and let $f : X \longrightarrow Y$ be any mapping from X to Y. Let A and B be subsets of X. Prove that $f(B) \setminus f(A) \subseteq f(B \setminus A)$ [3]

(c) Let X and Y be any sets, A be a subset of X, and let $f : X \longrightarrow Y$ be an injection from X to Y. Prove that $A = f^{-1}(f(A))$. [3]

(d) Let X and Y be any sets, and let $f : X \longrightarrow Y$ be a bijection from X to Y. Let A and B be subsets of Y. Prove that:

(i) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$; [3]

(ii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$; [3]

(iii) $f^{-1}(B) \setminus f^{-1}(A) = f^{-1}(B \setminus A)$. [3]

END OF EXAMINATION