UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2009/2010

B.Sc. / B.Ed. / B.A.S.S.III

TITLE OF PAPER	:	VECTOR ANALYSIS
COURSE NUMBER	:	M312
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) Find the outward unit normal vector to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a, b > 0,$$

at the point $P(\frac{-a}{\sqrt{2}}, \frac{b}{\sqrt{2}})$. [8]

(b) A curvilinear coordinate system (u, v, ϕ) is defined by

$$x = auv\cos\phi, \quad y = auv\sin\phi, \quad z = \frac{a}{2}(u^2 - v^2), \text{ where } u, v > 0,\ -\pi < \phi < \pi.$$

(i) Find the scale factors and the unit vectors. [6]

(ii) Show that the coordinate system is orthogonal. [3]

(iii) Find the line element and the volume element. [1,2]

QUESTION 2

(a) The path of a highway and an exit ramp are superimposed on a rectangular coordinate system in such a way that the highway coincides with the x-axis. The exit ramp begins at the origin O. After following the graph of $y = \frac{y^4}{4}$ from O to the point $P(1, \frac{-1}{4})$, the path follows the arc of a circle in such a way that the ramp is *continuous, smooth* and has *continuous curvature*. Find the equation of this circle. [10]

(b) In each of the following, find the arc length parameter along the curve from the point where $t = 0$ by evaluating the integral

$$s = \int_{\tau=0}^{t} |\mathbf{v}(\tau)| d\tau.$$

Then find the length of the indicated portion of the curve.

(i) $\mathbf{r}(t) = (4\cos t)\hat{\mathbf{i}} + (4\sin t)\hat{\mathbf{j}} + 3t\hat{\mathbf{k}}, \qquad 0 \le t \le \pi/2,$

(ii) $\mathbf{r}(t) = (e^t\cos t)\hat{\mathbf{i}} + (e^t\sin t)\hat{\mathbf{j}} + e^t\hat{\mathbf{k}}, \qquad -\ln 4 \le t \le 0.$ [10]

QUESTION 3

(a) Let $\mathbf{u}(x, y, z) = x\hat{\mathbf{i}} - y\hat{\mathbf{j}}$ and $\mathbf{v}(x, y, z) = \dfrac{\mathbf{u}}{(x^2 + y^2)^{\frac{1}{2}}}$ be vectors in space.

(i) Compute the divergence and the curl of $\mathbf{u}$ and $\mathbf{v}$. [1,1,1,2]

(ii) Find the flow lines of $\mathbf{u}$ and $\mathbf{v}$. [5,1]

(b) Determine the directional derivative of $\phi(x, y) = 100 - x^2 - y^2$ at the point (3,6) in the direction of the unit vector $\hat{\mathbf{u}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$. [3]

(c) Find the tangent plane and the normal line to the surface $x^2y + xyz - z^2 = 2$ at the point $P_0(1, 1, 3)$. [6]

QUESTION 4

(a) By any method, find the integral of $H(x, y, z) = yz$ over the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies above the cone $z = \sqrt{x^2 + y^2}$. [6]

(b) Find the work done in moving a particle in the counterclockwise direction once around the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ if the force field is given by $\mathbf{F} = (3x - 4y)\hat{\mathbf{i}} + (4x + 2y)\hat{\mathbf{j}} - 4y^2\hat{\mathbf{k}}$. [4]

(c) Find out which of the fields given below are conservative. For conservative fields, find a potential function.

(i) $\mathbf{F} = (yz^2)\hat{\mathbf{i}} + (xz^2)\hat{\mathbf{j}} + (x^2yz)\hat{\mathbf{k}}$. [2]

(ii) $\mathbf{F} = (e^x \sin y)\hat{\mathbf{i}} + (e^x \cos y + \sin z)\hat{\mathbf{j}} + (y \cos z)\hat{\mathbf{k}}$. [8]

QUESTION 5

(a) Evaluate $\iint_S [xz^2\,\mathrm{d}y\,\mathrm{d}z + (x^2y - z^3)\,\mathrm{d}z\,\mathrm{d}x + (2xy + y^2z)\,\mathrm{d}x\,\mathrm{d}y]$, where S is the entire surface of the hemispherical region bounded by $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$

(i) by the divergence theorem (Green's theorem in space), [5]

(ii) directly. [7]

(b) Verify Stokes' theorem for $\mathbf{A} = 3y\hat{\mathbf{i}} - xz\hat{\mathbf{j}} + yz^2\hat{\mathbf{k}}$, where S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$ and C is its boundary. [8]

QUESTION 6

(a) Evaluate $\iint_S \mathbf{A}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$, where $\mathbf{A} = xy\hat{\mathbf{i}} - x^2\hat{\mathbf{j}} + (x+z)\hat{\mathbf{k}}$, S is that portion of the plane $2x + 2y + z = 6$ included in the first octant, and $\hat{\mathbf{n}}$ is the unit normal to S. [10]

(b) Verify that the parametric equations

$$x = \rho^2\cos\theta, \quad y = \rho^2\sin\theta, \quad z = \rho$$

could be used to represent the surface $x^2 + y^2 - z^4 = 0$. Hence compute the unit normal to this surface at any point. [10]

QUESTION 7

Let $J_n(x)$ be the Bessel function of the first kind of order n.

(a) Evaluate

(a) $\displaystyle\int_0^1 \frac{\mathrm{d}x}{\sqrt{\ln \frac{1}{x}}}$. [3]

(b) $\displaystyle\int_0^\infty \sqrt{y}\, \exp^{-y^3}\, \mathrm{d}y$. [3]

(b) Express $J_4(ax)$ in terms of $J_0(ax)$ and $J_1(ax)$. [6]

(c) Legendre's differential equation is given by

$$(1 - x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0.$$

Using this, or by any other method, prove that

$$\int_{-1}^{1} P_m(x)P_n(x)\mathrm{d}x = 0, \quad \text{if } m \neq n.$$

[8]

END OF EXAMINATION