UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2009/2010

B.Sc. / B.Ed. / B.A.S.S.III

TITLE OF PAPER	:	VECTOR ANALYSIS
COURSE NUMBER	:	M312
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) (i) Find the distance from the plane $x+2y+6z=1$ to the plane $x+2y+6z=10$. [5]

(ii) Find parametric equations for the line through $(0,-7,0)$ perpendicular to the plane $x+2y+2z=13$. [5]

(b) (i) The graph $y=f(x)$ in the xy-plane automatically has the parametrization $x=x$, $y=f(x)$, and the vector formula $\mathbf{r}(x)=x\hat{\mathbf{i}}+(f(x))\hat{\mathbf{j}}$. Use this formula to show that if f is a twice differentiable function of x, then

$$\kappa(x)=\frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}.$$

[7]

(ii) Use the formula for κ in (i) to find the curvature of $y=\ln(\cos x)$, $-\pi/2\leq x\leq\pi/2$. [3]

QUESTION 2

(a) Show that $\mathbf{n}(t)=-g'(t)\hat{\mathbf{i}}+f'(t)\hat{\mathbf{j}}$ and $-\mathbf{n}(t)=g'(t)\hat{\mathbf{i}}-f'(t)\hat{\mathbf{j}}$ are both normals to the curve $\mathbf{r}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ at the point $(f(t),g(t))$. Hence find a unit normal, $\hat{\mathbf{N}}$, for the curve $\mathbf{r}(t)=\sqrt{4-t^2}\,\hat{\mathbf{i}}+t\hat{\mathbf{j}}$, $-2\leq t\leq 2$. [6]

(b) Integrate $f(x,y,z)=2x-6y^2+2z$ over the line segment C joining the points (2,2,2) and (3,3,3). [6]

(c) Show that $ydx+xdy+4dz$ is exact and evaluate the integral

$$\int_{(2,2,2)}^{(3,4,0)} ydx+xdy+4dz.$$

[8]

QUESTION 3

(a) Give a formula $\mathbf{F} = M(x,y)\hat{\imath} + N(x,y)\hat{\jmath}$ for the vector field in the plane with the properties that $\mathbf{F} = \mathbf{0}$ at the origin and that at any other point (a,b) in the plane, $\mathbf{F}$ is tangent to the circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction, with magnitude $|\mathbf{F}| = \sqrt{a^2 + b^2}$. [10]

(b) Verify the divergence theorem for $\mathbf{A} = (2x - z)\hat{\imath} + x^2 y\hat{\jmath} - xz^2\hat{\mathbf{k}}$ taken over the region bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. [10]

QUESTION 4

(a) By any method, find the outward flux of the field $\mathbf{F} = (6x^2 + 2xy)\hat{\imath} + (2y + x^2 z)\hat{\jmath} + (4x^2 y^3)\hat{\mathbf{k}}$ across the boundary of the region cut from the first octant by the cylinder $x^2 + y^2 = 9$ and the plane $z = 9$. [10]

(b) By any method, find the circulation of the field $\mathbf{F} = (x^2 + y^2)\hat{\imath} + (x + y)\hat{\jmath}$ around the triangle with vertices (1,0), (0,1), (-3,0) traversed in the counterclockwise direction. [10]

QUESTION 5

(a) If $\mathbf{F} = y\hat{\imath} + (x - 2xz)\hat{\jmath} - xy\hat{\mathbf{k}}$, evaluate $\iint_S (\nabla \times \mathbf{F})\cdot\hat{\mathbf{n}}\,\mathrm{d}S$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane. [10]

(b) Verify that the parametric equations

$$x = \rho^2 \cos\theta, \quad y = \rho^2 \sin\theta, \quad z = \rho$$

could be used to represent the surface $x^2 + y^2 - z^4 = 0$. Hence compute the unit normal to this surface at any point. [10]

QUESTION 6

(a) Part of a railway line (superimposed on a rectangular coordinate system) follows the line $y = -x$ for $x \leq 0$, then turns to reach the point (3,0) following a cubic curve. Find the equation of this curve if the track is *continuous*, *smooth*, and has *continuous curvature.* [10]

(b) Find the scale factors h_1, h_2, and h_3 in cylindrical and in spherical coordinates. Hence find the volume element dV (in cylindrical and in spherical coordinates). [10]

QUESTION 7

(a) Evaluate

(i) $\int x^4 J_1(x)\mathrm{d}x$ [2]

(ii) $\int x^3 J_3(x)\mathrm{d}x$ [4]

(iii) $\int_0^\infty \frac{y^2}{1+y^4}\mathrm{d}y.$ [3]

(iv) $\int_0^\pi \sin^5\theta\mathrm{d}\theta,$ [3]

where $J_1(x)$ and $J_3(x)$ are Bessel functions of the first kind of order 1 and order 3, respectively.

(b) Use recurrence relations to show that

$$2J_0''(x) + J_0(x) - J_2(x) = 0,$$

where $J_n(x)$ is the Bessel function of the first kind of order n. [8]

END OF EXAMINATION