University of Swaziland

Final Examination – May 2010

BSc III, Bass III, BEd III

Title of Paper : Complex Analysis

Course Number : M313

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

QUESTION 1

(a) Solve the equation $z^n = 1$, where n has any one of the values $n = 2, 3, \ldots$. What is the geometrical interpretation for $n = 2, 3, 4$? [8]

(b) Consider the complex plane. Give the definitions of

- (i) interior point, [1]
- (ii) open set, [1]
- (iii) connected set, [1]
- (iv) simply connected set. [1]

(c) (i) Sketch the following sets:

$$\arg(z+i) = \frac{3\pi}{4}, \qquad |z-2+i| \leq 1, \qquad \operatorname{Im} z > 1.$$

[2]

(ii) Which sets in (i) are domains? [1]

(iii) Which sets in (i) are bounded? [1]

(d) Construct the line $\operatorname{Re} \frac{1}{z} = 2$. [4]

QUESTION 2

(a) Find the region into which a transformation $w = f(z)$ maps a region D if

- (i) $f(z) = z^3$, D is the sector $z \leq 1,\ 0 \leq \theta \leq \pi/4$. [3]
- (ii) $f(z) = z + \frac{1}{z}$, D is the entire circle $|z| = 1$. [3]

(b) Find the limits. Give reasons for your solutions.

- (i) $\lim_{z \to i} \frac{iz + 2i}{z - i}$, [2]
- (ii) $\lim_{z \to \infty} \frac{z^2 + 1}{z - 1}$. [2]

(c) Explain the formula $\lim_{z \to z_0} f(z) = w_0$. [2]

(d) Using just the definition of the derivative, find $f'(z)$ for the following:

- (i) $f(z) = |z|^2$. Use $z\bar{z} = |z|^2$. [6]
- (ii) $f(z) = \bar{z}$. [2]

QUESTION 3

(a) (i) State,

(ii) and prove the necessary conditions theorem for the existence of $f'(z_0)$. [1,5]

(b) Use Cauchy-Riemann equations to show that $f'(z)$ does not exist if $f(z) = \exp(x)\exp(-iy)$. [3]

(c) Use the sufficient conditions theorem to show that $f'(z)$ and is derivative $f''(z)$ exist everywhere, and find $f''(z)$ when $f(z) = \exp(x)\exp(-iy)$. [3]

(d) Derive the formula

$$f'(z) = \exp(-i\theta)(u_r + iv_r)$$

in the usual notations. [8]

QUESTION 4

(a) Define

(i) Analytic function, [1]

(ii) Entire function, [1]

(iii) Singular point, [1]

(iv) Harmonic function, [1]

(v) Harmonic conjugate of u. [1]

(b) Consider $f(z) = |z|^2$.

(i) Is $f(z)$ analytic? Explain. [2]

(ii) Find singular points. [2]

(c) Find the analytic function $w = f(z)$, given that the real part $u(x, y) = 2e^x \cos y$ and $f(0) = 2$. [7]

(d) Show that if v is a harmonic conjugate of u in a domain D, then $-u$ is a harmonic conjugate of v in D, and cinversely. [4]

QUESTION 5

(a) Expand $\frac{1}{z^2+2z}$ into the simple fractions and thus show that

$$\int_C \frac{dz}{z^2+2z} = i\pi,$$

where C is a positively oriented cirlce $|z| = 1$. [8]

(b) State the Cauchy-Goursat theorem. [2]

(c) Apply the Cauchy integral formula to show that

$$\int_C \frac{z\,dz}{(9-z^2)(z+i)} = \frac{\pi}{5},$$

where C is the positively oriented circle $|z| = 2$. [10]

QUESTION 6

(a) State the Taylor's theorem for analytic functions. [2]

(b) (i) Prove that

$$\sin z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}, \qquad |z| < \infty.$$

[2]

(ii) Show that

$$\sinh z = -i\sin(it),$$

[2]

(iii) and hence derive

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \qquad |z| < \infty.$$

[2]

(c) (i) Find the Laurent series representation for $\exp(1/z)$, [6]

(ii) and hence prove that

$$\int_C \exp(1/z)\,dz = 2\pi i,$$

where C is any positively oriented simple closed contour around the origin. [6]

QUESTION 7

(a) Give the definitions and examples of

(i) Isolated singular point, [2]

(ii) Residue, [2]

(iii) Principal part of $f(z)$ at z_0, [2]

(iv) Pole of order m. [2]

(b) Using the residue theorem, evaluate the following integrals:

(i) $\int_C \frac{5z-2}{z(z-1)}\,dz$, where C is the circle $|z|=2$ described counterclockwise, [4]

(ii) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$. [8]

END OF EXAMINATION PAPER