University of Swaziland

Supplementary Examination – July 2010

BSc IV, BASS IV, BEd IV

Title of Paper : Complex Analysis
Course Number : M313
Time Allowed : Three (3) hours
Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

QUESTION 1

(a) Find the roots of the equation $z^3 + 8i = 0$. [8]

(b) Consider the complex plane. Give the definitions of

(i) exterior point, [1]
(ii) closed set, [1]
(iii) bounded set, [1]
(iv) multiply connected set. [1]

(c) (i) Sketch the following sets:

$$|z + 2i| = 2, \qquad \operatorname{Im} z < 1, \qquad \operatorname{Re} z = 2.$$

[2]

(ii) Which sets in (i) are domains? [1]
(iii) Which sets in (i) are bounded? [1]

(d) Construct the line $\operatorname{Im} \dfrac{1}{z+i} = 1$. [4]

QUESTION 2

(a) Find the region into which a transformation $w = f(z)$ maps a region D if

(i) $f(z) = z^4$, D is the sector $r \leq 1,\ 0 \leq \theta \leq \pi/4$. [3]
(ii) $f(z) = z^2$, D is the square $0 \leq \operatorname{Re} z \leq 1,\ \ 0 \leq \operatorname{Im} z \leq 1$. [3]

(b) Find the limits. Give reasons for your solutions.

(i) $\displaystyle\lim_{z\to -2} \frac{iz+2}{z+2}$, [2]
(ii) $\displaystyle\lim_{z\to\infty} \frac{2z+i}{z+1}$. [2]

(c) Define the function $f(z)$ continuous at a point z_0. [2]

(d) Using just the definition of the derivative, find $f'(z)$ for the following:

(i) $f(z) = |z|^2$. Use $z\bar{z} = |z|^2$. [6]
(ii) $f(z) = \operatorname{Im} z$. [2]

QUESTION 3

(a) Using Cauchy-Riemann equations

(i) state a sufficient conditions theorem for existence of $f'(z_0)$, [1]

(ii) check if there are derivatives of $f(z) = z^2$ and $g(z) = |z|^2$, so find $f'(z)$ and $g'(z)$. [6]

(b) Use Cauchy-Riemann equations to show that $f'(z)$ does not exist if $f(z) = z - \bar{z}$. [3]

(c) Use the sufficient conditions theorem to show that $f'(z)$ and is derivative $f''(z)$ exist everywhere, and find $f''(z)$ when $f(z) = \cos x \cosh y - i \sin x \sinh y$. [4]

(d) Derive Cauchy-Riemann equations in polar coordinates. [6]

QUESTION 4

(a) Define

(i) Analytic function, [1]

(ii) Entire function, [1]

(iii) Singular point, [1]

(iv) Harmonic function, [1]

(v) Harmonic conjugate of u. [1]

(b) Consider $f(z) = \dfrac{1}{z}$.

(i) Is $f(z)$ analytic? Explain. [2]

(ii) Find singular points. [2]

(c) Given $u(x, y) = e^x[x \cos y - y \sin y]$.

(i) Find f as an explicit function of z, where the real part of $f(z)$ is u. [8]

(ii) Find $f'(z)$. [3]

QUESTION 5

(a) Show that

(i) $\int_C \frac{dz}{z-a} = 2i\pi$ where C is a cirlce $|z-a| = R$. [4]

(ii) $\int_C e^{-z}\, dz = 1$, where C is a ray, $\arg z = \pi/4$. [5]

(b) (i) State, and [1]

(ii) prove the Cauchy integral formula. [4]

(c) Apply the Cauchy integral formula to evaluate

$$\int_C \frac{e^{2iz}}{z^2+2z}\, dz,$$

where C is the circle $|z| = 1$. [6]

QUESTION 6

(a) State the Laurent series theorem. [2]

(b) (i) Expand $f(z) = \frac{1}{(z-1)^2(z-3)}$ in a Laurent series valid for $0 < |z-1| < 2$. [8]

(ii) What is the principal part of the series in (i)? [1]

(c) (i) Prove that

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad |z| < \infty,$$

[3]

(ii) show that $\cosh z = \cos(iz)$, [3]

(iii) and hence derive a Taylor series for $\cosh z$. [3]

QUESTION 7

(a) Give the definitions and examples of

(i) Isolated singular point, [2]

(ii) Residue, [2]

(iii) Principal part of $f(z)$ at z_0, [2]

(iv) Pole of order m. [2]

(b) (i) Derive the formula

$$\int_C f(z)\, dz = 2\pi i b_1$$

in the usual notations. Explain the terms. [2]

(ii) Use the results from (i) to evaluate

$$\int_C \frac{e^{-z}}{(z-1)^2}\, dz,$$

where C is the circle $|z| = 2$, described in the positive sense. [4]

(c) Using the residue theorem, evaluate

$$\int_0^\infty \frac{\cos x}{x^2 + b^2}\, dx, \quad b > 0.$$

[6]

END OF EXAMINATION PAPER