University of Swaziland

Final Examination, 2009/10

BSc III, Bass III, BEd III

Title of Paper : Abstract Algebra I

Course Number : M323

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Show that if $(a, m) = 1$ and $(b, m) = 1$, then $(ab, m) = 1$ where $a, b, m \in \mathbb{Z}$. [4]

(b) Prove that the binomial coefficient $_pC_r = \begin{pmatrix} p \\ r \end{pmatrix}$ with $0 < r < p$ is divisible by the positive prime p. [4]

(c) Use (b) to show that when $a, b, p \in \mathbb{N}$, p prime, then

$$(a + b)^p \equiv a^p + b^p \pmod{p}. \qquad [4]$$

(d) Prove that a group of prime order has no proper subgroup. [4]

(e) Prove that every group of prime order is cyclic. [4]

Question 2

(a) Let G be the set of all 2×2 matrices of the form $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ where $a, b, c \in \mathbb{Q}$, $ac \neq 0$.
Show that, with respect to matrix maultiplication, G is a group. [8]

(b) Solve the system

$$\begin{aligned} 3x &\equiv 2 \pmod{5} \\ 2x &\equiv 1 \pmod{3}. \end{aligned}$$

[8]

(c) Give an example of a group satisfying the given conditions or, if there is no such group, say so (Do not prove anything).

(i) A finite non-abelian group

(ii) A non-abelian cyclic group.

[4]

Question 3

(a) Let n be a positive integer greater than 1 and let, for $a, b \in \mathbb{Z}$

$$aRb \Longleftrightarrow a \equiv b \pmod{n}.$$

Prove that R is an equivalence relation on $\mathbb{Z}$. [5]

(b) Prove that every cyclic group is abelian. [5]

(c) Let H be the subset

$$\{\rho_0 = (1), \rho_1 = (1\,2\,3), \rho_2 = (1\,3\,2)\}$$

of symmetric group S_3.

(i) Show that H is a subgroup of S_3. [5]

(ii) Show that H is cyclic. [5]

Question 4

Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 1 & 7 & 5 & 3 & 4 & 2 \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 1 & 8 & 3 & 4 & 5 & 2 & 6 \end{pmatrix}.$$

(a) Express α and β as products of disjoint cycles, and then as products of transpositions. For each of them, say whether it is an even permutation or an odd one, [7]

(b) Compute α^{-1}, $\beta^{-1}\alpha$, $(\alpha\beta)^{-1}$. [7]

(c) Find the order of β and compute β^{2010}. [6]

Question 5

(a)

(i) State Cayley's theorem [do not prove anything] [4]

(ii) Let $(\mathbb{R}, \bullet)$ be the multiplicative group of all positive real numbers and $(\mathbb{R}, +)$ be the additive group of real numbers.
Show that $(\mathbb{R}, \bullet)$ is isomorphic to $(\mathbb{R}, +)$. [6]

(b)

(i) Find the number of generators in each of the following cyclic groups $\mathbb{Z}_{30}$ and $\mathbb{Z}_{42}$. [6]

(ii) Determine the cosets of $H = <4>$ in $\mathbb{Z} + 8$. [4]

Question 6

(a) Solve the following system

$$\begin{aligned} 2x &\equiv 1 \pmod 5 \\ 3x &\equiv 4 \pmod 7). \end{aligned}$$

[7]

(b) Determine whether the given set G with respect to the given operation is a group.

(i) Define $*$ on $G = \mathbb{Q}^+$ by $a * b = \frac{1}{2}ab,\ \forall\ a, b \in G = \mathbb{Q}^+$. [5]

(ii) Define $*$ on $G = \mathbb{R}$ by $a * b = ab + a + b\ \forall\ a, b \in G = \mathbb{R}$. [5]

(c) Show that $\mathbb{Z}_6$ and S_3 are not isomorphic. [3]

Question 7

(a) Find the number of elements in each of the cyclic subgroups [do not list the elements]

(i) $<30>$ of $\mathbb{Z}_{42}$ [3]

(ii) $<15>$ of $\mathbb{Z}_{48}$ [3]

(b) For $\mathbb{Z}_{12}$, find all subgroups and give a lattice diagram. [7]

(c) State Lagrange's theorem [do not prove]. [2]

(d) Using (c) above, or otherwise, show that $\mathbb{Z}_p$ has no proper subgroups if p is a prime number. [5]