University of Swaziland

Supplementary Examination, 2009/10

BSc III, Bass III, BEd III

Title of Paper : Abstract Algebra I

Course Number : M323

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Suppose that d, a, b are positive integers, $(a, d) = 1$ and that d divides ab. Proves that d divides b. [5]

(b) Let H be a subgroup of a group G and let, for $a, b \in G$

$$aRb \text{ if and only if } a = g^{-1}bg \text{ for some } g \in H.$$

Show that R is an equivalence relation on the set G.[6]

(c) The table below may be completed to define a binary operation $*$ on the set $G = \{e, a, b, c\}$ in such a way that $(G, *)$ becomes a group. Asuume this is possible and compute the missing entries

$*$	e	a	b	c
e	e	a	b	c
a	a	e		b
b	b		a	
c	c	b		a

[6]

(d) Prove that, in any group G, the identity element is unique. [3]

Question 2

(a) Determine whether the set $\mathbb{Z}$ with respect to the binary operation

$$a * b = a + b - 2010$$

is a group. [7]

(b) Find the greatest common divisor d of the numbers 102 and 42 and express it in the form

$$d = 102m + 42n \text{ for some } m, n \in \mathbb{Z}.$$ [5]

(c)

(i) State Lagrange's Theorem [2]

(ii) Prove that every finite group of prime order is cyclic. [6]

Question 3

(a) Let H be the subset

$$\{\rho_0 = (1), \rho_1 = (1\,2\,3\,4), \rho_2 = (1\,3)(2\,4), \rho_3 = (1\,4\,3\,2)\}$$

of symmetric group D_4.

(i) Show that H is a subgroup of D_4. [5]

(ii) Is H cyclic? Justify your answer. [5]

(b) Let $\phi G \Longrightarrow H$ be an isomorphism of groups.

(i) Prove that, if e_g is the identity element of G, then $(e_g)\phi$ is the identity element of H.

(ii) Prove that, for any $a \in G$,

$$(a^{-1})\phi = \left[(a)\phi\right]^{-1}.$$ [6]

(c) Determine all possible solutions of

$$3x \equiv 5 (\text{mod } 11).$$ [4]

Question 4

(a) Prove that a non-abelian group of order $2p$, p prime, contains at least one element of order p. [6]

(b) Consider the following permutations in S_6

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 3 \end{pmatrix}.$$

Compute

(i) $\alpha\beta$ (ii) β^2 (iii) β^{-2}

(iv) $\alpha\beta^2$ (v) $(\alpha\beta)^{-1}$ [10]

(c) Write the permutations α and β in (b) as a product of disjoint cycles in S_6. [4]

Question 5

(a) Prove that every subgroup of a cyclic group is cyclic. [8]

(b) Let H be the subgroup of $\mathbb{Z}_{20}$ generated by the element 8, i.e. $H = <8>$. Find all cosets of H in $\mathbb{Z}_{20}$. [6]

(c) Prove that if G is a group and that $\forall\ a \in G, a^2 = e$, then G is abelian. [6]

Question 6

(a) Prove that every group of prime order is cyclic. [5]

(b) Show that the set $G = Q - \{0\}$ with respect to the operation

$$a * b = \frac{ab}{10}$$

is a group. [9]

(c) Prove that if $(ab)^{-1} = a^{-1}b^{-1}$ for every $a, b \in G$ where G is a group, then G is abelian. [6]

Question 7

(a) Find all subgroups of $\mathbb{Z}_{20}$ and draw a lattice diagram. [8]

(b)

(i) Define *a subgroup of a group.*

(ii) Find the number of elements in the cyclic subgroup $<30>$ of $\mathbb{Z}_{42}$ (do not list the elements).

(d) Show that $\mathbb{R}$ under addition is isomorphic to $\mathbb{R}^+$ under multiplication. [5]